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Using bigraphs to model topological graphs embedded in orientable surfaces

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ABSTRACT

Natural and artificial environments, at scales ranging from cellular to geographic, have complex and changing spatial structures based on regions, as well as being inhabited by a multiplicity of dynamic entities. Milner's theory of bigraphs provides a formal design tool for dynamic and complex systems. However, bigraphs have rather limited explicit capability to represent spatial properties and relationships, being only equipped with a place graph that can express the containment relation between locations. This paper develops constructions that provide explicit bigraph types for representing complex two-dimensional spatial configurations, and shows that such representations are unique up to topological equivalence. In particular, we show how bigraphs can uniquely represent topological graphs embedded in compact, orientable surfaces in $\Re^n$, as well as in the Euclidean plane.

1. Introduction

Natural and artificial environments, from cellular to geographic scales, have complex, dynamic, region-based structures, with the component regions inhabited by a multiplicity of dynamic entity and sensor types. At the cellular level, membrane computing aims to model structure and function in terms of dynamic collections of regions, each providing a space in which reactions can take place (e.g., [1,2]). At the geographic scale, there are models (e.g., [3]) of ubiquitous computing in the form of spatially embedded mobile devices, monitoring and responding to complex, dynamic, real world events. In such models space plays an essential role, and the models need to be expressive enough to represent the richness of spatial entities and relationships.

Milner, in a series of papers [4–6], developed the theory of bigraphs to support the formal modeling of complex and dynamic systems. Bigraphs originated in process calculi, especially the calculus of mobile ambients [7], and the π-calculus [8,9], both of which provide formal models of mobility and connectivity. Bigraphs provide a formal method for independently specifying connectivity and locality, and are intended to provide an intuitive representation of both virtual and physical systems. A bigraph consists of a common set of nodes structured by a *place graph* (a forest of trees) and a *link graph* (a hypergraph, where each edge can connect more than two nodes). The nodes therefore can stand for situated entities (represented by the place graph) which can form connections with each other (represented by the link graph). In short, a bigraph is a mathematical model of two key real world concepts – location and connection – and can thus represent in a basic way the situation of entities in the world.

The spatial relationship expressed by the place graph is that of containment, in that node m is a descendant of node n in the place graph if the entity represented by m is inside the entity represented by n. The containment relation has rather limited power on its own to express the wide range of spatial structures, properties, and relations that exist in the world. Our objective is to use the structures within the bigraph model itself to allow a bigraph to explicitly represent richer kinds

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of spatial structures. In particular, we focus on representations of the topology of the modeled space and show that bigraphs have the power to uniquely represent a wide class of two-dimensional spatial configurations embedded in certain surfaces. The topological representation modeled here goes beyond containment to include adjacency and connectivity. We are able to show that the bigraph model developed is able to uniquely represent connected and non-connected topological graphs, up to homeomorphism. The benefit of this approach is that we can now use bigraphs to not only represent static topological structures (using the combinatorial map that in now explicitly part of the bigraph machinery) but also the dynamics that can be formally expressed in bigraphical reaction rules. Therefore, bigraphs enhance combinatorial maps by introducing formal specification of topological dynamics.

2. Surfaces, graph embeddings, and combinatorial maps

Our overall objective is to construct bigraphs that can uniquely represent the topology of graph embeddings in certain surfaces. So, before we go any further, we need to state the kinds of surfaces under consideration, and define an embedding of a graph in such a surface.

The surfaces that are considered in this paper are the compact, orientable surfaces in $\Re^n$. A surface in $\Re^n$ is compact if, and only if, the surface is closed and bounded. The classification theorem for orientable surfaces (first proved by Möbius, 1870) tells us that any compact, orientable surface in $\Re^n$ is homeomorphic to a sphere with m handles (m-holed torus), for $m \geq 0$.

Definition 1. An *embedding* of a graph in a surface is an injective map of the vertices and edges of a graph into the surface, in which vertices are mapped to points on the surface, and edges onto simple arcs (homeomorphic images of the closed real unit interval) on the surface, subject to the conditions that:

1. The end vertices of an edge are mapped to endpoints of the corresponding arc.
2. Two arcs can only intersect at their endpoints.
3. An arc cannot contain points associated with other vertices.

Informally, an embedding of a graph into a surface is a drawing of the graph on the surface in such a way that its edges may intersect only at their endpoints.

When we consider the compact, orientable surfaces in $\Re^n$, we have the nice result that the complement in the surface of a graph embedding gives us a collection of regions, and each of these regions will be a 2-manifold, locally homeomorphic to a disk. If, furthermore, each of the faces is homeomorphic to a disc, the embedding is called a 2-*cell embedding*.

We may note in passing that only connected graphs can admit 2-cell embeddings, and that the first part of this paper concerns only embeddings of connected graphs.

The surfaces in which we realize spatial structures should allow the provision of a clear method of uniquely characterizing a region in terms of its boundary, and therefore of having an unambiguous notion of "insideness". For the Euclidean plane $\Re^2$, the Jordan Curve Theorem tells us that the complement of the image of a simple closed arc embedded in $\Re^2$ consists of two connected components, each disconnected from the other. One of these components is bounded (the interior) and the other is unbounded (the exterior). So, for the plane, there is a clear sense in which an entity is inside an areal object, assuming that the areal object is bounded by a simple closed arc. However, for the compact, orientable surfaces in $\Re^n$, there is no unambiguous notion of "insideness". Nevertheless, if we consider the arc to be directed, then it is possible to uniquely define the enclosed region. (We will use the convention that this region is the one we see on the left as we traverse the arc in its positive direction.)

So in summary, the kinds of surfaces that we allow embeddings of graphs into are orientable 2-manifolds for which it is possible to uniquely define regions enclosed by simple closed arcs within them and homeomorphic to discs.

There is one final construction that is needed for this development, and that is that of a combinatorial map.

Definition 2. A *combinatorial map* consists of:

1. A finite set S of semi-edges (also called darts)
2. A permutation γ of S
3. An involutory permutation β of S with no fixed point

subject to the constraint that the permutation group generated by γ and β is transitive.

Intuitively, a combinatorial map provides a representation of a 2-cell graph embedding, where each arc is represented by a pair of semi-edges, called *facing edges*, transposed by β. Each cycle of γ represents an arc endpoint, and the cycle specifies a counter-clockwise cyclic ordering of arcs specified by its semi-edges around endpoints.

It is noteworthy that the composition $\alpha = \beta\gamma^{-1}$ defines the ordering of semi-edges around each face of the graph. Each face is defined as the region on the left while traversing the semi-edges of a cycle of α. In fact, α, β, and γ are dependent, and any pair of them can be used to compute the third, and therefore represent the combinatorial map.

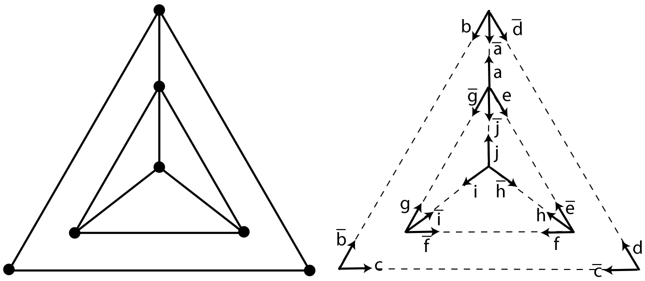

Fig. 1. Combinatorial map example.

In a combinatorial map, semi-edges are denoted by a, $\bar{a}$, x, $\bar{x}$, etc., with the convention that $x\beta$ is denoted $\bar{x}$. Also, permutations are usually written in cycle form, where $(a_0 \dots a_{n-1})$ is the notation for the permutation cycle $a_i \to a_{i+1 \bmod n}$. Fig. 1 shows an example of topological graph and its combinatorial map, with its semi-edges labeled according to the scheme:

$$\gamma = (a\overline{gj}e)(b\overline{ad})(c\bar{b})(\bar{c}d)(\bar{e}hf)(\bar{f}ig)(\bar{h}ji)$$
$$\beta = (a\bar{a}) \cdots (j\bar{j}).$$

We observe that $\alpha = (abcd\bar{a}efg)(\bar{f}hi)(\overline{gij})(\overline{bdc})(\overline{ejh})$ defines the five faces (including the "outer" face) of the figure.

The important thing about a combinatorial map is that it is an algebraic structure that uniquely characterizes the topological characteristics of the associated graph embedding. Indeed, we have the following theorem, due to Edmonds [11] and Tutte [10].

Theorem 1. *Each combinatorial map provides a topologically unique (up to homeomorphism of the surficial embeddings) representation of a* 2*-cell graph embedding in a closed surface. Conversely, every* 2*-cell graph embedding in a closed surface can be uniquely (up to permutation group isomorphism) represented by a combinatorial map.*

It is possible to generalize a combinatorial map to higher dimensions, but we shall not need that here.

3. Bigraphs

3.1. Introduction and terminology

Rather than repeat Milner's definitions, we instead present the features of bigraphs needed for this work by means of examples, and refer the reader to [6] for details. Fig. 2 gives two examples of simple, concrete bigraphs, labeled **Ex1** and **Ex2**. (All the bigraphs in this paper will be concrete, and from now on we omit the term.) A bigraph is composed of two graphs: a *place graph* and a *link graph*, shown for bigraph **Ex1** in the left and right parts of Fig. 3, respectively. The place graph is a forest, and represents the containment of places one inside another (e.g., v_1 is contained in v_0). The link graph is a hypergraph, where hyperedges may join more than two places (note, for example, the hyperedge e in Fig. 3), and represents connections between places. A place may be a *node* (as with v_0, v_1, v_2 of **Ex1**), a *root* (as with the roots 0 and 1 of the place graph of **Ex1**), or a *site* (as with leaf 0 of the place graph of **Ex1**). In the diagram, a site is represented as a shaded region with a hatched line boundary, and a root as an unshaded region with a hatched line boundary. Nodes may have *ports* which allow links to other places. They are indicated in Fig. 2 as small, filled circles. Nodes are classified using *controls* (as with control K in **Ex1** and **Ex2**), which provide a simple typing system, specifying numbers of ports for each type of node.

A bigraph has an *interface* comprising its *inner face* of sites and *inner names* and its *outer face* of roots and *outer names*. Inner and outer names provide names of dangling edges in incomplete hyperedges, thus providing ways of combining link graphs of two or more bigraphs (see later for bigraph composition). The convention is that inner and outer names are at the bottom and top, respectively, of a bigraph diagram. For example, the inner face of bigraph **Ex1**, comprises site 0 and inner name x, and the outer face comprises roots 0, 1, and outer names y_1, y_2. We can think of a bigraph as a function from its inner face to its outer face. For bigraph **Ex1**, we can write:

$$\mathbf{Ex1} : \langle \{0\}, \{x\} \rangle \to \langle \{0, 1\}, \{y_1, y_2\} \rangle$$

and similarly for bigraph **Ex2**, we can write:

$$\mathbf{Ex2} : \langle \emptyset, \emptyset \rangle \to \langle \{0\}, \{x\} \rangle .$$

3.2. Some special bigraphs

Now we introduce some specific bigraphs to be used later. The first is the *identity bigraph*, $\mathbf{id}_{\vec{x}} : \langle 0, \vec{x} \rangle \to \langle 0, \vec{x} \rangle$, where $\vec{x}$ denotes the ordered n-tuple $(x_1, \dots, x_n)$. The bigraph $\mathbf{id}_{\vec{x}}$ is shown at the upper left of Fig. 4 and consists only of links between inner and outer names. (Note that in Fig. 4 most node and edge names have been omitted for clarity.)

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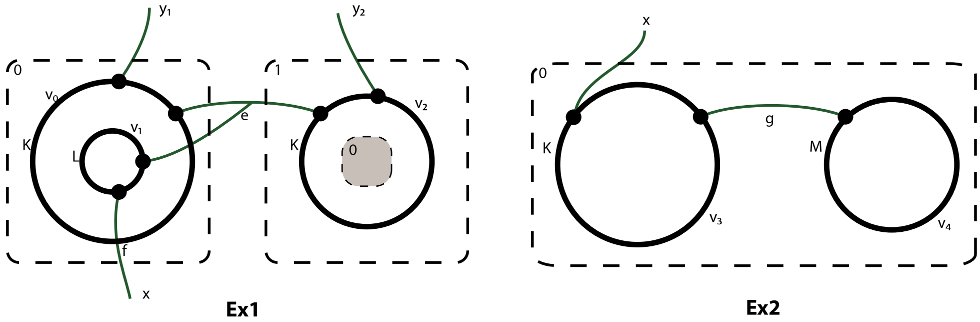

Fig. 2. Bigraphs **Ex1** and **Ex2**.

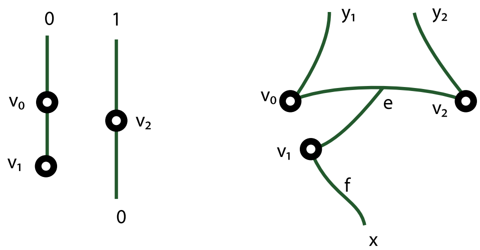

Fig. 3. The place and link graphs associated with the bigraph **Ex1** from Fig. 2.

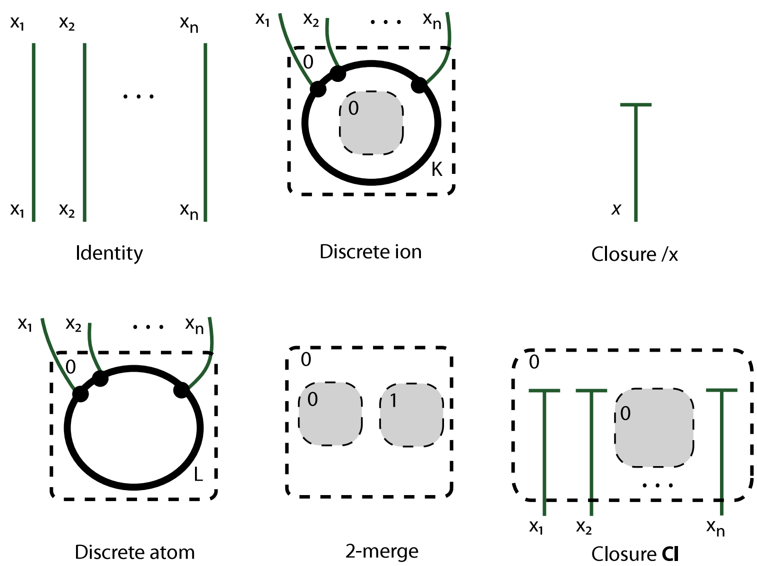

Fig. 4. Some basic bigraphs.

We also need some building block bigraphs, from which larger edifices are constructed. A *discrete ion* is a bigraph having a single node linking to a collection of zero or more distinct outer names, and a single site and root. The upper middle of Fig. 4 shows the discrete ion $\mathbf{K}_{\vec{x}} : \langle\{0\}, \emptyset\rangle \to \left\langle\{0\}, \vec{x}\right\rangle$.

A *discrete atom* is a bigraph with a single node linking to a collection of zero or more distinct outer names, with no sites and a single root. The lower left of Fig. 4 shows the discrete atom $\mathbf{L}_{\vec{x}} : \langle\emptyset, \emptyset\rangle \to \left\langle\{0\}, \vec{x}\right\rangle$.

Another basic bigraph, to be used later in this paper, is

$$\mathbf{merge}_n : \langle\{0, \ldots, n-1\}, \emptyset\rangle \to \langle\{0\}, \emptyset\rangle$$

$\mathbf{merge}_n$ is a node-free bigraph with no links, n sites and one root. By way of example, $\mathbf{merge}_2$ is shown at the lower middle of Fig. 4.

Finally, we introduce two forms of the *closure* bigraph, used to eliminate open links. The first form is a node-free graph with no places, and just a collection of inner names. If $\vec{x}$ are the inner names, then the closure bigraph is denoted $/\vec{x} : \langle 0, \vec{x}\rangle \to \langle 0, \emptyset\rangle$. The closure bigraph $/x$ is shown in the upper right part of Fig. 4.

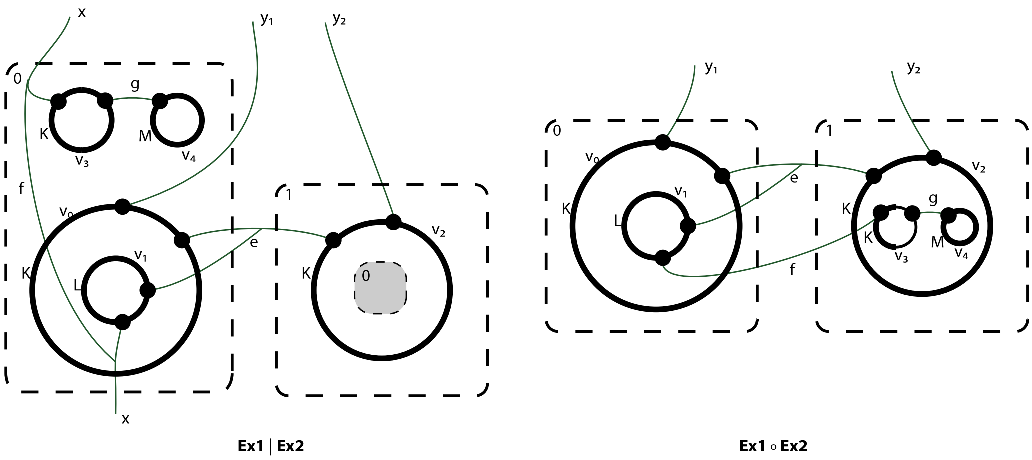

Fig. 5. Merge and composition of bigraphs **Ex1** and **Ex2**.

In what follows it will be useful to close off links while preserving sites and roots. We introduce the closure bigraph $\mathbf{Cl}_{\vec{x}} : \langle 1, \vec{x} \rangle \to \langle 1, \emptyset \rangle$ shown diagrammatically in the lower right part of Fig. 4.

3.3. Combining bigraphs

There are three methods for combining bigraphs used in this paper: forming a tensor product, merging, and composition. In the first case, assume we have bigraphs **X** and **Y** that have disjoint sets of nodes, edges, inner and outer names. Then the *tensor product* of the bigraphs, written $\mathbf{X} \otimes \mathbf{Y}$, is formed by simply juxtaposing the two link graphs and place graphs (with some renumbering of roots and sites as required). For example, the closure bigraph $\mathbf{Cl}_{\vec{x}}$ defined in the previous subsection may be defined as the tensor product $/\vec{x} \otimes \mathbf{merge}_1$.

This can be generalized to the case where **X** and **Y** have disjoint sets of nodes and edges, but may share inner and outer names. (There are other constraints, but we omit them in this brief discussion, and all merges in what follows satisfy these constraints.) Then, the *merge* of **X** and **Y**, written **X**|**Y**, is formed by constructing the unions of corresponding link and place graphs where, in the new link graph, links with the same name are combined. In this operation, we also bring together roots with the same name. The merge operation is associative. The left-hand part of Fig. 5 shows the merge **Ex1**|**Ex2**. We note that the 0-roots of the bigraphs have been merged, and inner name x connected to outer name x through edge f.

Finally, there is a composition operation that mirrors function composition. We stated earlier that a bigraph can be thought of as a function from its inner face to its outer face. If we assume that the inner face of **X** is equal to the outer face of **Y**, then we may form the *composition*, $\mathbf{X} \circ \mathbf{Y}$, by merging these two faces together, as follows:

- In the place graph identify in order each root of **Y** with the corresponding site of **X**.
- In the link graph join together a link of the outer face of **Y** to a link of the inner face of **X** if the name of the former is equal to the name of the latter. The linking names then disappear.

The right-hand part of Fig. 5 shows the composition $\mathbf{Ex1} \circ \mathbf{Ex2}$. The constituents of **Ex2** are now contained in node v_2 of **Ex1**, and the outer name x of **Ex2** is connected to the inner name x of **Ex1** through edge f. The linking name x, having done its job, disappears. Expressed in function form, the interface of $\mathbf{Ex1} \circ \mathbf{Ex2}$ is:

$$\mathbf{Ex1} \circ \mathbf{Ex2} : \langle \emptyset, \emptyset \rangle \to \langle \{0, 1\}, \{y_1, y_2\} \rangle \,.$$

4. Fundamental spatial constructions

This section constructs bigraphs needed to formalize the spatial configurations of interest. (Note that Milner uses the term "point" differently from our geometric interpretation below.)

4.1. Semi-points

The basic zero-dimensional object is the semi-point (s-point), so called because in the interpretation in the next section it needs to pair with another s-point to be interpreted as an embedded point. We define two kinds of semi-points: begin-s-point and end-s-point, necessary to provide orientation to the interpreted structures.

Definition 3. A *begin-s-point* is a discrete atom $\mathbf{B}_x$ with a single outer name x. An *end-s-point* is a discrete atom $\mathbf{E}_x$ with a single outer name x.

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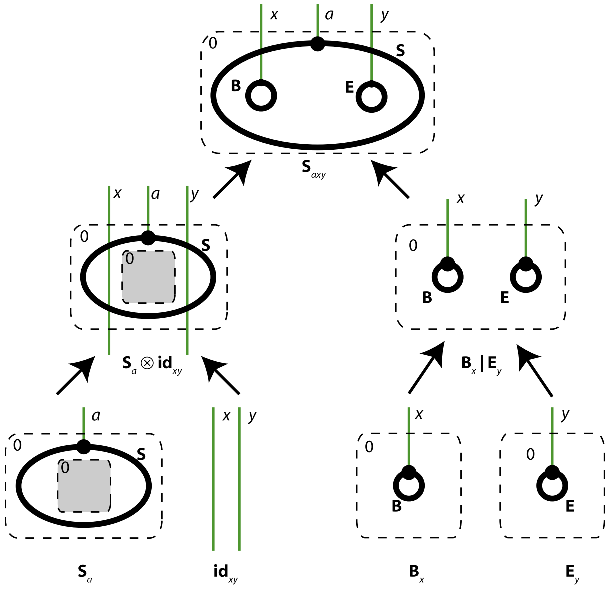

Fig. 6. Examples of a begin s-point and an end s-point, and their amalgamation to a semi-edge.

The outer names forming part of the begin-s-point and end-s-point bigraphs act to label the bigraphs, and often we refer to these bigraphs just by these names.

4.2. Semi-edges

Begin and end s-points are used as components semi-edges. As with s-points, semi-edges work in pairs to make embedded edges in the interpretation given later. Their construction as bigraphs is given by the following definition, shown schematically in Fig. 6.

Definition 4. A *semi-edge* $\mathbf{S}_{axy}$ is defined as:

$$\mathbf{S}_{axy} = (\mathbf{S}_a \otimes \mathbf{id}_{xy}) \circ (\mathbf{B}_x | \mathbf{E}_y)$$

where $\mathbf{S}_a$ is a discrete ion with a single outer name a. We say that $\mathbf{S}_{axy}$ is the semi-edge a with begin s-point x and end s-point y.

We impose some additional uniqueness constraints.

Constraint 1. If $\mathbf{S}_{axy}$ and $\mathbf{S}_{bxy}$ are semi-edges having the same begin s-point, x, and the same end s-point, y, then $a = b$ and the semi-edges are identical.

Constraint 2. Given begin and end s-points, x, y, respectively. Then there is at most one semi-edge $\mathbf{S}_{axy}$ with begin s-point x and end s-point y.

Therefore, the semi-edge name uniquely determines its begin and end s-points, and this allows us to simplify our notation. As in most of what follows, we will be focusing on edges, rather than points, we will often use the shorthand $\mathbf{S}_a$ to denote the semi-edge $\mathbf{S}_{axy}$, in the knowledge that the unique x and y can be retrieved when needed.

4.3. Semi-edge combination and cycles

The way that semi-edges relate to one another is through their constituent s-points.

Definition 5. Let $\mathbf{S}_{axy}$ and $\mathbf{S}_{byz}$ be two semi-edges, where the end s-point of $\mathbf{S}_a$ has the same name as the begin s-point of $\mathbf{S}_b$. Then $\mathbf{S}_{axy}$ and $\mathbf{S}_{byz}$ are said to form a *connected ordered pair*.

Note that the ordering is important here. Just because $\mathbf{S}_a$ and $\mathbf{S}_b$ form a connected ordered pair does not imply that $\mathbf{S}_b$ and $\mathbf{S}_a$ form such an ordered pair. If the connection relation goes both ways, we have the following definition:

Definition 6. Let $\mathbf{S}_{axy}$ and $\mathbf{S}_{byx}$ two semi-edges, where the end s-point of $\mathbf{S}_a$ has the same name as the begin s-point of $\mathbf{S}_b$, and the end s-point of $\mathbf{S}_b$ has the same name as the begin s-point of $\mathbf{S}_a$. Then $\mathbf{S}_{axy}$ and $\mathbf{S}_{byx}$ are said to form a *facing pair*.

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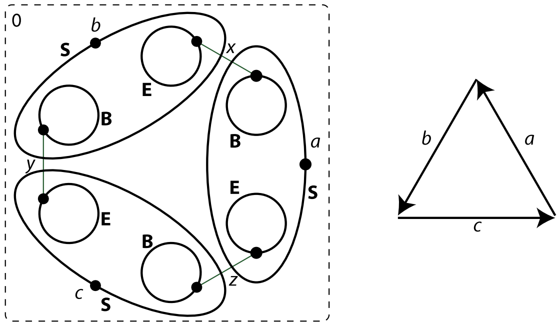

Fig. 7. The cycle $\mathbf{C}_{abc}$ and its skeleton abc.

It is clear that if $\mathbf{S}_a$ and $\mathbf{S}_b$ form a connected ordered pair and $\mathbf{S}_b$ and $\mathbf{S}_a$ form a connected ordered pair, then $\mathbf{S}_a$ and $\mathbf{S}_b$ form a facing pair. It is also clear that if $\mathbf{S}_a$ and $\mathbf{S}_b$ form a facing pair, then $\mathbf{S}_b$ and $\mathbf{S}_a$ form a facing pair. We often use the notation $\mathbf{S}_{\overline{a}}$ for the semi-edge facing semi-edge $\mathbf{S}_a$, if such a semi-edge exists.

Appropriately related semi-edges can be amalgamated into cyclical structures as follows.

Definition 7. Let $\mathbf{S}_{a_1}, \ldots, \mathbf{S}_{a_n}$ be n pairwise distinct semi-edges, where the semi-edges form a collection of connected ordered pairs, $\mathbf{S}_{a_1}$ and $\mathbf{S}_{a_2}$, $\mathbf{S}_{a_2}$ and $\mathbf{S}_{a_3}, \ldots, \mathbf{S}_{a_n}$ and $\mathbf{S}_{a_1}$. Then a *cycle bigraph* $\mathbf{C}_{\vec{a}}$ is defined as:

$$\mathbf{C}_{\vec{a}} = \mathbf{Cl}_{\vec{a}\vec{x}} \circ (\mathbf{S}_{a_1} | \cdots | \mathbf{S}_{a_n}).$$

The number n of semi-edges in a cycle bigraph is termed its *order*. Clearly, $\mathbf{C}_{a_1 a_2, \ldots a_n} \equiv \mathbf{C}_{a_2 \ldots a_n a_1} \equiv \cdots$

In pure geometric terms, the outer names of s-points and semi-edges are no longer needed, as no further compositions are required. In application domains where these and later geometric entities are given further semantic significance (e.g., a cellular membrane or a wall of a room), then they can be used to embed geometries into other structures. However, to simplify the constructions, we will close off all outer names from now on. We can still use vectors of the closed off semi-edge outer names to name these structures. To simplify the diagrams, we do not show closed off links originating from a single port, but show the link name at the port from which the link originated. When there is no ambiguity, we often refer to a cycle bigraph as a cycle. The left-hand side of Fig. 7 shows an example of the cycle $\mathbf{C}_{abc}$ of order 3, formed from semi-edges $\mathbf{S}_a$, $\mathbf{S}_b$, and $\mathbf{S}_c$.

Definition 8. Let $\mathbf{C}_{\vec{a}}$ be a cycle bigraph. The *skeleton* of $\mathbf{C}_{\vec{a}}$ is a directed graph whose nodes are the links connecting end and begin s-points of a connected pair of semi-edges, and the edges are the semi-edges, labeled by the names of the edges, with direction given by the direction from end to begin s-point. Edges connect to nodes in the obvious way.

The right-hand side of Fig. 7 shows the skeleton of cycle $\mathbf{C}_{abc}$. The structure of a cycle bigraph can be completely captured from its skeleton. Although there is a restriction imposed on a cycle that constituent semi-edges are pairwise disjoint, there is no restriction on the presence of facing pairs of semi-edges, or multiple occurrences of s-points. We will see later how these are interpreted as embeddings in orientable surfaces.

4.4. Polygons

Most existing representations of the topology of spatial objects are based upon boundary representations, derived from Baumgart's winged-edge structure [12,13], and the quad-edge structure of Guibas and Stolfi [14]. However, it is often regions rather than their edges, that are more directly appropriate for conceptualizations of real-world spaces. While edges are important, often representing the boundaries of features (e.g., geopolitical borders, building walls, and cell membranes), it is often the features themselves that are primary, and the boundaries secondary. In domains that go beyond pure graphical representations, we need to associate semantics with domain entities, and in many cases the associations are more naturally with the regions themselves than with their boundaries. In this subsection, we explicitly represent the region that a cycleset bounds.

Definition 9. Let $\mathbf{P}$ be a discrete ion with no outer names, and $\mathbf{C}_{\vec{a}}$ be a cycle. An *n-site polygon*, $\mathbf{P}^n_{\vec{a}}$ is defined for $n \geq 0$ as:

$$\mathbf{P}^n_{\vec{a}} = \mathbf{P} \circ (\mathbf{C}_{\vec{a}} | \mathbf{merge}_n).$$

For the purely geometric purposes of this account, ion $\mathbf{P}$ was given no outer names, but when modeling a specific domain we may want to add outer names to attach assets to the area. Fig. 8 shows the 2-site polygon $\mathbf{P}^2_{abc}$ that is an extension of the cycle $\mathbf{C}_{abc}$ shown in Fig. 7. When $n = 0$ the definition simplifies to $\mathbf{P}_{\vec{a}} = \mathbf{P} \circ \mathbf{C}_{\vec{a}}$, and such a polygon is referred to as a *ground polygon*. For ground polygons we omit the superscript. For the remainder of this section, and the next on surficial embeddings, all the polygons will be ground. We shall return to non-ground polygons later, when discussion non-connected structures.

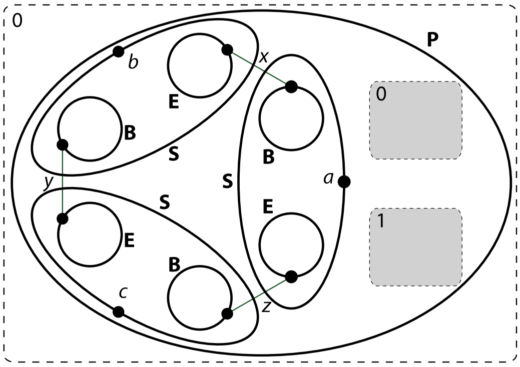

Fig. 8. The 2-site polygon $\mathbf{P}^2_{abc}$.

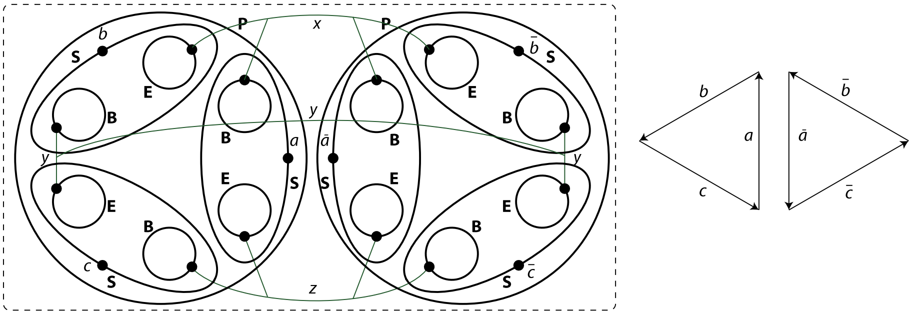

Fig. 9. A map and its skeleton.

4.5. Map bigraphs

Cycles may be aggregated by "stitching together" any pairs of facing edges that they share. But before we present the definition, we need to ensure that the aggregation is of a piece. To this end, we introduce the notion of a permutation group associated with a set of cycles. There is an obvious sense in which a single cycle bigraph may be viewed as a cyclic permutation of its semi-edges. We extend this as follows:

Definition 10. Let $\mathbf{T}$ be the set of semi-edges in the cycles $\mathbf{C}_{\vec{a}^1}, \ldots, \mathbf{C}_{\vec{a}^k}$, where all constituent semi-edges in the total collection of cycles are pairwise distinct and form a complete set of facing pairs. Then, define two permutations on the combined set of semi-edges as follows:

- $\alpha(\mathbf{T})$ is the composition (as permutations) of the cyclic permutations coming from each of its constituent cycle bigraphs.
- $\beta(\mathbf{T})$ is the involutory permutation that transposes each semi-edge with its facing semi-edge.

Define $\mathcal{G}(\mathbf{T})$ to be the permutation group generated by $\alpha(\mathbf{T})$ and $\beta(\mathbf{T})$. If $\mathcal{G}(\mathbf{T})$ is transitive as a permutation group on the semi-edges, then we say that $\mathbf{T}$ is *connected*.

We are now ready to define the principal spatial construction of this paper.

Definition 11. Let $\mathbf{P}_{\vec{a}^1}, \ldots, \mathbf{P}_{\vec{a}^k}$ be a collection of k (ground) polygons. Let $\mathbf{T}$ be the set of semi-edges in the cycles $\mathbf{C}_{\vec{a}^1}, \ldots, \mathbf{C}_{\vec{a}^k}$, where all constituent semi-edges in the total collection of cycles are pairwise distinct and form a complete set of facing pairs. We stipulate further that $\mathbf{T}$ is to be connected. Then a *map bigraph* (or *map*) $\mathbf{M}$ is defined as:

$$\mathbf{M} = \mathbf{P}_{\vec{a}^1} | \cdots | \mathbf{P}_{\vec{a}^k}.$$

Given a map $\mathbf{M}$, we will slightly abuse notation and refer to the two permutations defined in Definition 10 as $\alpha(\mathbf{M})$ and $\beta(\mathbf{M})$. The skeleton of a map is the union of the collection of skeletons of its individual cycles. Fig. 9 shows the map with cycles $\alpha(\mathbf{M}) = (abc)(\overline{cba})$ and $\beta(\mathbf{M}) = (a\bar{a})(b\bar{b})(c\bar{c})$. It is clear that the semi-edges form a complete set of facing edges. It is also easy to check that the generated group is transitive on the six semi-edges, being in fact the symmetric group on three elements.

5. Embedding a map bigraph in a compact, orientable surface in $\Re^n$

In this section we develop a condition for the embedding of a map bigraph in a compact, orientable surface in $\Re^n$, and show that any such embedding, if it exists, will be unique up to topological equivalence. As we saw earlier, a surface in $\Re^n$

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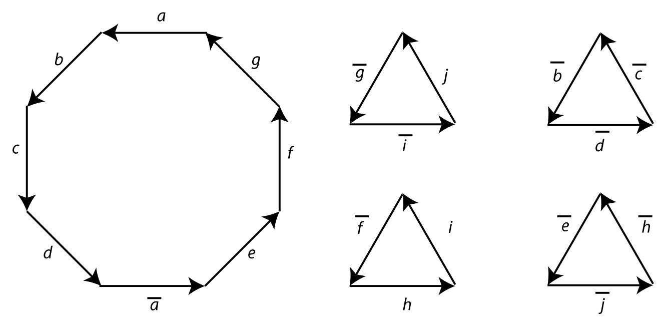

Fig. 10. Map **Ex3**.

is compact if, and only if, the surface is closed and bounded, and the compact, orientable surfaces in $\Re^n$ are homeomorphic to spheres with m handles ($m \geq 0$). We also saw that Edmonds [11] and Tutte [10] showed that each combinatorial map represents a topologically unique 2-cell graph embedding in a compact, orientable surface. Therefore, all we need to show is that every map bigraph can be represented as a combinatorial map, and our result will directly follow.

Let **M** be a map bigraph. Then in addition to the two permutations, $\alpha(\mathbf{M})$ and $\beta(\mathbf{M})$ on its set of semi-edges, we may define a third

$$\gamma(\mathbf{M}) = \beta(\mathbf{M})\alpha(\mathbf{M})^{-1}.$$

A little thought convinces us that $\gamma(\mathbf{H})$ gives the cycles of the semi-edges associated with each begin s-point in the representation of **H**. This is all that is needed to use the results of Edmonds and Tutte to show that the representation of the map bigraph **A** as a 2-cell embedding in a compact, orientable surface is a combinatorial map and therefore unique up to topological equivalence.

The embedding itself may be constructed as follows. Each skeleton cycle determines the boundary of a region. Imagine that we are traversing a skeleton cycle in the direction of the cycle's directed semi-edges. Then the region associated with a cycle is the region on our left as we make our traversal. The individual regions associated with the map bigraph's polygons may now be fitted together like a jigsaw, matching each facing pair of semi-edges. As the map bigraph is connected, so is the graph structure formed by the facing pairs of semi-edges. Because in the map bigraph every semi-edge is matched with another, the entire surface is covered by the embedded bigraph.

For example, Fig. 10 shows the map **Ex3** defined by:

$$\mathbf{Ex3} = \mathbf{C}_{abcd\bar{a}efg}|\mathbf{C}_{\bar{f}hi}|\mathbf{C}_{\overline{gi}j}|\mathbf{C}_{\overline{bdc}}|\mathbf{C}_{\overline{ejh}}.$$

Here,

$$\alpha(\mathbf{Ex3}) = (abcd\bar{a}efg)(\bar{f}hi)(\overline{gi}j)(\overline{bdc})(\overline{ejh})$$
$$\beta(\mathbf{Ex3}) = (a\bar{a}) \cdots (j\bar{j})$$
$$\gamma(\mathbf{Ex3}) = (a\overline{gj}f)(b\overline{ad})(c\bar{b})(\bar{c}d)(\bar{e}hf)(\overline{fi}g)(\bar{h}ji).$$

That **Ex3** is a map can be seen by checking that the constituent semi-edges form a complete facing set and that the generated group is transitive. Fig. 11 shows an embedding of **Ex3** on the surface of the sphere. This embedding is unique, up to topological equivalence.

In order to determine in which surface a particular map bigraph is embeddable, we need some definitions and a construction for combining map cycles together (different from permutation composition used above). If **M** is a map bigraph, then the constituent cycles of $\alpha(\mathbf{M})$ are termed the *cycles* of **M**.

Definition 12. Let $X = (a_1 \dots a_n)$ and $Y = (b_1 \dots b_m)$ be two cycles, such that the following two conditions hold.

1. Each b_j is an a_i
2. If b' is an immediate successor to b in the cyclic ordering of Y, then there is no $b'' \in Y$ between b and b' in the cyclic ordering of X.

Then we say that Y is a *subcycle* of X.

Definition 13. Let $X = (a_1 a_2 \dots a_n)$ be a cycle of map **M**. Then X is an *n-handle* ($n \geq 0$) if it contains exactly n pairwise disjoint sub-cycles $(a_1 b_1 \bar{a}_1 \bar{b}_1), \dots, (a_n b_n \bar{a}_n \bar{b}_n)$.

Definition 14. Let X and Y be two cycles of map **M**. Then a *gluing* of X and Y is defined as the cycle formed by identifying a facing pair of edges and following around the cycle in order. More formally, let $X = (a_1 a_2, \dots a_m)$ and $Y = (b_1 b_2 \dots b_n)$, where a_i and b_j form a facing pair. Then the *gluing* of X and Y using a_i and b_j is the cycle:

$$(a_1 \dots a_{i-1} b_{j+1} \dots b_n b_1 \dots b_{j-1} a_{i+1} \dots a_m).$$

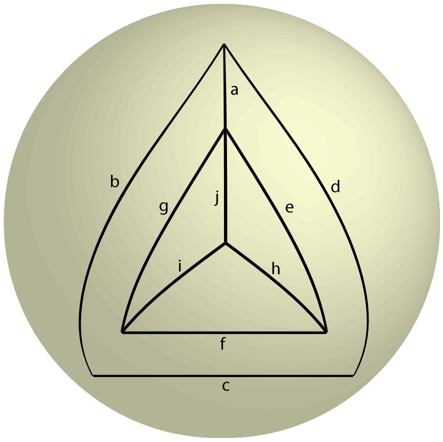

Fig. 11. Embedding of **Ex3**.

For example, ($abcdefg$) glued to ($\bar{g}ij\bar{d}klh$) by identifying the facing pair g and $\bar{g}$ is the cycle ($abcdefij\bar{d}klh$). Note that gluing is a symmetric operation, provided the same facing pair of edges is chosen.

We will use gluing and the notion of handles to give a condition for a map bigraph to be embeddable on the surface of a sphere. We need one last construction.

Definition 15. Let **M** be a map bigraph. A *gluing* of **M** is the cycle resulting from any gluing together of its cycles, beginning with a single pair with a common facing edge, and continuing to glue pairs of faces together. We note that because $\mathcal{G}(\mathbf{M})$ is transitive, we are able to glue on faces until the result is a single cycle.

A standard result from algebraic topology [15] allows us to make a connection between the number of handles that a compact surface has and the "algebraic" handle as given in the above definition. So, we are able to immediately state the general result as follows:

Proposition 1. *Let **M** be a map bigraph, and let $G(\mathbf{M})$ be a gluing of **M**. Then **M** is embeddable on the surface of the n-holed torus if and only if $G(\mathbf{M})$ is an n-handle.*

Note that when $n = 0$, the result specializes to the condition for embedding a map bigraph in the surface of a sphere, and we state this special case explicitly as a corollary.

Corollary 1. *Let **M** be a map bigraph, and let $G(\mathbf{M})$ be a gluing of **M**. Then **M** is embeddable on the surface of the sphere if and only if $G(\mathbf{M})$ is a 0-handle.*

Returning to our example map bigraph, **Ex3**, we start to glue cycles together, starting from the left, and generating successively:

$$(abcd\bar{a}ehig),\ (abcd\bar{a}ehi\bar{i}j),\ (a\bar{d}\bar{c}cd\bar{a}ehi\bar{i}j),\ (a\bar{d}\bar{c}cd\bar{a}\bar{j}\bar{h}hi\bar{i}j).$$

A check on the final expression soon convinces us that $G(\mathbf{Ex3})$ is a 0-handle, and so **Ex3** is embeddable in the surface of a sphere, as shown in Fig. 11.

The reader may wonder whether it matters which particular gluing we choose. The above proposition allows us to immediately deduce the following corollary.

Corollary 2. *Let **M** be a map bigraph, and let $G_1(\mathbf{M})$ and $G_2(\mathbf{M})$ be two gluings of **M**. Suppose $G_1(\mathbf{M})$ is an n_1-handle and $G_2(\mathbf{M})$ is an n_2-handle, for some $n_1, n_2 \geq 0$. Then $n_1 = n_2$.*

To see this, note that by Proposition 1, **M** at the same time is embeddable in and covers the surface of an n_1-holed torus and is embeddable in and covers the surface of an n_2-holed torus. Because the embedding determines the number of handles the surface contains, we must have $n_1 = n_2$.

To summarize the constructions of this section, we have shown that every map bigraph provides a topologically unique representation of a 2-cell graph embedding in some surface. We can encapsulate this result more precisely in the following theorem.

Theorem 2. *Let **M** be a map bigraph. Then there exists an algorithm, to transform **M** into a 2-cell graph embedding $\mathcal{G}(\mathbf{M})$. Conversely, if **G** is a 2-cell embedding, then there exists an algorithm to transform **G** into a map bigraph $\mathcal{M}(\mathbf{G})$. Furthermore, $\mathcal{G}\mathcal{M}(\mathbf{G}) = \mathbf{G}$ modulo homeomorphism of surficial embeddings, and $\mathcal{M}\mathcal{G}(\mathbf{M}) = \mathbf{M}$ modulo bigraph element relabelings.*

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The proof of this result is contained in the constructions of this section. However, we can spell out explicitly some details. Let **M** be a map bigraph. We have shown that $\langle\alpha(\mathbf{M}), \beta(\mathbf{M}), \gamma(\mathbf{M})\rangle$ is transitive as a permutation group and thus a combinatorial map. Theorem 1 now gives the 2-cell embedding (up to homeomorphism of the embedding). Conversely, given 2-cell embedding, and hence any combinatorial map (unique up to permutation group isomorphism), a bigraph can be constructed for which that map is a representation, as we now show. Let combinatorial map **M** be defined by its two characteristic permutations of semi-edges, $\gamma(\mathbf{M})$ giving the cycles (in anti-clockwise order) of the semi-edges associated with each vertex of **M**, and the involutory permutation $\alpha(\mathbf{M})$ giving the facing pairs of semi-edges. Construct the permutation $\beta(\mathbf{M})$ as the composition $\alpha(\mathbf{M})\gamma(\mathbf{M})$. This gives the cycles (in anti-clockwise order) of semi-edges around regions, with regions on the left as the cycles are traversed. We may use these cycles to construct the skeleton cycles of a map bigraph, and hence the bigraph itself. Finally, it is clear that each of these two constructions is the inverse of the other, modulo relabeling and appropriate morphisms. Hence the result follows.

6. Non-connected areas

Up to now, the spatial entities have been connected topological graphs, whose faces cover and partition the surfaces in which they are embedded. When we remove the constraint that the graphs are connected, we have to add some extra structure so that bigraphs can continue to represent the topological structure. Bigraph composition will represent the notion of one area being in the interior of another. However, a difficulty arises with maps as defined so far. Up to now it has been unnecessary to specify a map boundary, because when embedded in a closed orientable surface, there is no distinguished boundary — the map covers the entire surface. However, when we wish to place a map inside a polygon, then the boundary of the map that immediately abuts the polygon area must be specified if we wish to represent the topology of the structure uniquely. The notion of a map with boundary is developed next.

6.1. Maps with boundary

By designating a single polygon to be the exterior of the area, we arrive at the notion of an area with boundary. To this end, we define a variant of the polygon bigraph that we term the *exterior polygon bigraph*. It's definition is as follows.

Definition 16. Let **X** be a discrete ion with no outer names, and $\mathbf{C}_{\vec{a}}$ be a cycle. An *exterior polygon*, $\mathbf{X}_{\vec{a}}$ is defined as:

$$\mathbf{X}_{\vec{a}} = \mathbf{X} \circ \mathbf{C}_{\vec{a}}.$$

So, an exterior polygon has the same structure as a ground polygon, but is distinguished by a node of kind **X**. Exterior polygons allow us to define regions with boundary, as follows.

Definition 17. Let $\mathbf{P}_{\vec{a}_1}, \ldots, \mathbf{P}_{\vec{a}_k}$ be a collection of k polygons and $\mathbf{X}_{\vec{a}}$ be an exterior polygon, where all constituent semi-edges in the total collection of $k+1$ cycles are pairwise distinct and form a complete set of facing pairs. Let **T** be the set of semi-edges in all the constituent cycles, and suppose that **T** is connected. Then a *bounded map bigraph* **N** is defined as:

$$\mathbf{N} = \mathbf{P}_{\vec{a}^1} | \cdots | \mathbf{P}_{\vec{a}^k} | \mathbf{X}_{\vec{a}}.$$

The expected interpretation as an embedding holds here. When situated inside an embedded polygon, the bounded area bigraph presents its external face outwards toward the bounding polygon. This constraint allows the bigraph structure to continue to represent the topology of the embedding uniquely.

6.2. Non-ground maps

In Section 4 we introduced the notion of the n-site polygon, which has spaces in which further spatial structures can be composed. In this section, we generalize the notion of a map to allow non-ground polygons as components.

Definition 18. Let $\mathbf{P}^{n_1}_{\vec{a}_1}, \ldots, \mathbf{P}^{n_k}_{\vec{a}_k}$ be a collection of k n_i-site polygons ($1 \le i \le k$, $n_k \ge 0$), where $n > 0$ is the sum of all the n_i and all constituent semi-edges in the total collection of k cycles are pairwise distinct and form a complete set of facing pairs. Let **T** be the set of semi-edges in all the constituent cycles, and suppose that **T** is connected. Then a *non-ground map bigraph* $\mathbf{M}^n$ is defined as:

$$\mathbf{M}^n = \mathbf{P}^{n_1}_{\vec{a}^1} | \cdots | \mathbf{P}^{n_k}_{\vec{a}^k}.$$

In a similar manner, we can generalize the notion of a bounded map to a non-ground, bounded map bigraph.

Definition 19. Let $\mathbf{P}^{n_1}_{\vec{a}_1}, \ldots, \mathbf{P}^{n_k}_{\vec{a}_k}$ be a collection of k n_i-site polygons ($1 \le i \le k$, $n_i \ge 0$), and suppose that $n > 0$ is the sum of all the n_i. Let $\mathbf{X}_{\vec{a}}$ be an exterior polygon. Suppose that all constituent semi-edges in the total collection of $k+1$ cycles are pairwise distinct and form a complete set of facing pairs. Let **T** be the set of semi-edges in all the constituent cycles, and suppose that **T** is connected. Then a *non-ground, bounded map bigraph* $\mathbf{M}^n$ is defined as:

$$\mathbf{M}^n = \mathbf{P}^{n_1}_{\vec{a}^1} | \cdots | \mathbf{P}^{n_k}_{\vec{a}^k} | \mathbf{X}_{\vec{a}}.$$

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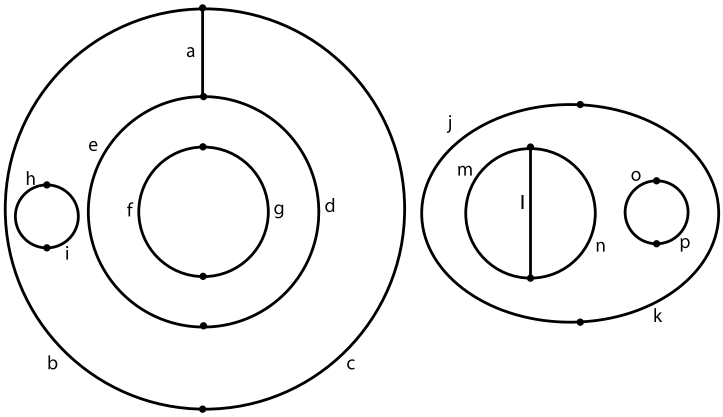

Fig. 12. Embedding of the scene **S**.

6.3. Scene bigraphs

We are now ready to situate maps inside maps in an embedding, the resulting bigraphs representing the non-connected embeddings being termed *scene bigraphs*. Suppose, in an embedding, we want to place map $\mathbf{M}_1$ inside map $\mathbf{M}_2$. It is clear that $\mathbf{M}_1$ will need to be situated in one of the constituent polygons of $\mathbf{M}_2$. The interior map must therefore be structured as a bounded map bigraph for the reasons given above. So, in general, apart from the "outer" map, all component maps will be bounded map bigraphs.

An example will illustrate this. Consider the embedding onto the surface of a sphere shown in Fig. 12. This is an embedding with five components, each one a connected topological graph. Because we are on a closed surface, it is a matter of choice (depending on semantics beyond the geometry) as to which we take to be the containing surface. In our example, we take the component in which all the other components are contained to be represented by the non-ground map bigraph **M** with three sites, as defined below:

$$\mathbf{M}^3 = \mathbf{P}^1_{abc\bar{a}de} | \mathbf{P}^1_{\overline{de}} | \mathbf{P}^1_{\overline{bc}}.$$

The three internal components of $\mathbf{M}^3$ are represented by bounded map bigraphs $\mathbf{N}_1$, $\mathbf{N}_2$, and $\mathbf{N}_2$, given below.

$$\mathbf{N}_1 = \mathbf{P}_{hi} | \mathbf{X}_{\overline{hi}}$$
$$\mathbf{N}_2 = \mathbf{P}_{fg} | \mathbf{X}_{\overline{fg}}$$
$$\mathbf{N}^2_3 = \mathbf{P}^2_{jk} | \mathbf{X}_{\overline{jk}}.$$

Bounded maps $\mathbf{N}_1$ and $\mathbf{N}_2$ have no further internal components and are ground. (Note that all the non-superscripted polygons and maps are ground.) Non-ground bounded map $\mathbf{N}^2_3$ has itself two internal components:

$$\mathbf{N}_4 = \mathbf{P}_{lm} | \mathbf{P}_{\bar{l}n} | \mathbf{X}_{\overline{mn}}$$
$$\mathbf{N}_5 = \mathbf{P}_{op} | \mathbf{X}_{\overline{op}}.$$

The scene is constructed in stages. Firstly we construct the bounded scenes, that is those bounded by polygons. These are $\mathbf{T}_1 = \mathbf{N}_1$, $\mathbf{T}_2 = \mathbf{N}_2$, and:

$$\mathbf{T}_3 = \mathbf{N}^2_3 \circ (\mathbf{N}_4 \otimes \mathbf{N}_5).$$

Now the composite scene can be constructed as:

$$\mathbf{S} = \mathbf{M}^3 \circ (\mathbf{T}_1 \otimes \mathbf{T}_2 \otimes \mathbf{T}_3).$$

Note the importance of order (both merge and tensor product are non-commutative).

We are now ready to give the general definition of a *scene*, preceded by that of a *bounded scene*.

Definition 20. The set of bounded scene bigraphs is the least set $\mathcal{T}$ of bigraphs that satisfies

1. All (ground) bounded maps are members of $\mathcal{T}$.
2. Let $\mathbf{N}^n$ be a non-ground, bounded map and $\{\mathbf{T}_1, \ldots, \mathbf{T}_n\} \subset \mathcal{T}$, where also that the sets of constituent semi-edges of each of the scenes are pairwise disjoint. Then

$$\mathbf{T} = \mathbf{N}^n \circ (\mathbf{T}_1 \otimes \cdots \otimes \mathbf{T}_n) \in \mathcal{T}.$$

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Definition 21. The set of scene bigraphs is the least set $\mathcal{S}$ of bigraphs that satisfies

1. All (ground) maps are members of $\mathcal{S}$.
2. Let $\mathbf{N}^n$ be a non-ground map and $\{\mathbf{T}_1, \ldots, \mathbf{T}_n\} \subset \mathcal{T}$, where the sets of constituent semi-edges of each of the scenes are pairwise disjoint. Then,

$$\mathbf{S} = \mathbf{N}^n \circ (\mathbf{T}_1 \otimes \cdots \otimes \mathbf{T}_n) \in \mathcal{S}.$$

Each of the connected component maps of a scene represents uniquely an embedded topological graph. The definitions in this section ensure that, given the choice of the containing bigraph, the insideness relationships between components is uniquely (up to topological equivalence) represented by the scene bigraph construction. (We should note that there is a certain degree of flexibility in the ordering of the elements in the scene bigraph, and for a precise uniqueness result we would need to quotient this out.)

As for the number of holes in which the scene is to be embedded, we saw in the previous section that each connected component can be embedded in the surface of an n-holed torus. The number of holes in the body on whose surface a scene is embedded is the sum of the numbers of holes in the bodies on whose surfaces are embedded each of the scene's constituent maps.

6.4. Planar embeddings

The Euclidean plane $\Re^2$ is not a compact surface, and so not covered in the above discussion. However, in practice planar embeddings are often the most important and, because the outside of a map embedded in a bigraph is uniquely defined, result in unique embeddings. The Euclidean plane $\Re^2$ is topologically equivalent to a punctured sphere. Suppose we have a map bigraph that is embeddable on the surface of a sphere. We now puncture the sphere, taking care that the puncture does not lie on one of the arcs of the embedding. (The cases where the puncture lies on an arc or node does not concern us, as we are not concerned with unbounded embeddings in the plane.) We now have a planar embedding of the map bigraph. We can think of the face in which the puncture occurs as being the (infinite) bounding face. Therefore, a collection of topological graphs embedded in the plane can be represented by a bounded scene. For example, consider the graph shown in Fig. 12 embedded in the plane. This can be represented by the planar scene :

$$\mathbf{PS} = \mathbf{T}_1 \otimes \mathbf{T}_2 \otimes \mathbf{T}_3.$$

The topological nature of a planar embedding is dependent upon which face the puncture occurs. By redefining constituent bounded maps to have different external faces, we will get different planar embeddings.

7. Conclusion and related work

This paper has demonstrated how Milner's bigraph structures can be used to represent connected and non-connected combinatorial graphs embedded in orientable surfaces, and that these representations are unique up to homeomorphism. Bigraphs have inbuilt in their place graphs the notions of container and contained, but do not have any other way of directly representing spatial relations. However, we have shown by construction how their link graphs may be utilized to provide bigraphs with the power to represent a large class of two-dimensional objects, namely the connected and non-connected topological graphs, and furthermore that the representation is unique up to topological equivalence. So bigraph theory does provide a formal framework in which spatial properties and relationships can be well expressed. These structures can be used to model complex spatial configurations, from cellular structures to the complexities of indoor spaces. Using bigraph operations, the spatial structures developed here can be enhanced with semantic information to provide rich domain representations.

Of course, all the structures discussed here are two-dimensional, being embedded in surfaces. If this work is to become truly useful for three-dimensional domains, such as can be found in systems biology, it needs to be extended to a third spatial dimension. We can extend the notion of a combinatorial map to a 3-dimensional combinatorial map using the following definition. (In fact, the extension can be made to any finite number of dimensions [16].)

Definition 22. A *3-map* consists of:

1. A finite set S
2. A permutation γ of S
3. Involutory permutations β_1 and β_2 of S with no fixed point, such that $\gamma\beta_1$ is an involution.

It can be shown that a 3-map, up to topological equivalence, uniquely represents the subdivision of a closed, orientable 3-dimensional space.

Regarding the temporal dimension, an important motivation for using bigraphs is their capability to handle spatial change. Milner's intent in presenting the bigraph theory was for it to be used as a formal design tool for ubiquitous computing involving distributed, mobile processes. If this goal is to be realized in the context of spatial computation, the work needs to be taken further. Previous work by the author [17] has used process calculi to represent types of spatial change, but without the full power to develop representations of topological structures that this paper provides. Work on extensions of the π calculus that can represent affine transformations is the 3π system presented in [18], and for Euclidean spaces the SpacePi system in [19] that restricts process communication to a fixed ball within Euclidean space. Although both systems have some topological capability, in particular being able to represent the topology of containment, they are not concerned with the topology of adjacency and connectivity, discussed in this paper. Other work [20] has characterized types of topological change, and developed decentralized algorithms for detecting such changes, using wireless sensor networks. There is currently a lot of interest in using process calculi to formally represent spatio-structural changes at the cellular level (e.g., in membranes [1,21,22]). The bigraph structures developed here provide the possibility of a powerful formalism for representing change, being able to capture both the richness of topological structure and process representations. In fact, bigraphs have recently begun to be used to formally model aspects of cellular interactions (e.g., [2,23,24]). This is an area of ongoing research.

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References

[1] G. Paun, G. Rozenberg, A guide to membrane computing, Theoretical Computer Science 287 (1) (2002) 73–100. http://dx.doi.org/10.1016/S0304-3975(02)00136-6.
[2] G. Bacci, D. Grohmann, M. Miculan, Bigraphical models for protein and membrane interactions, in: V. Danos, V. Schachter (Eds.), Proc. MeCBIC, 2009, pp. 3–18.
[3] C. Becker, F. Durr, On location models for ubiquitous computing, Personal and Ubiquitous Computing 9 (1) (2004) 20–31. http://dx.doi.org/10.1007/s00779-004-0270-2.
[4] R. Milner, Axioms for bigraphical structure, Mathematical Structures in Computer Science 15 (2005) 1005–1032.
[5] R. Milner, Pure bigraphs: Structure and dynamics, Information and Computing 204 (1) (2006) 60–122.
[6] R. Milner, The Space and Motion of Communicating Agents, Cambridge University Press, 2009.
[7] L. Cardelli, A.D. Gordon, Mobile ambients, in: Foundations of Software Science and Computation Structures: First International Conference, FOSSACS'98, in: Lecture Notes in Computer Science, Springer-Verlag, Berlin, Germany, 1998.
[8] R. Milner, Communicating and Mobile Systems: The π-calculus, Cambridge University Press, 1999.
[9] D. Sangiorgi, D. Walker, PI-Calculus: A Theory of Mobile Processes, Cambridge University Press, New York, NY, USA, 2001.
[10] W. Tutte, What is a map? in: New Directions in the Theory of Graphs, Academic Press, New York, 1973, pp. 309–325.
[11] J. Edmonds, A combinatorial representation for polyhedral surfaces, Notices of the American Mathematical Society 7 (1960) 646.
[12] B. Baumgart, Winged edge polyhedron representation, Tech. Rep. CS-TR-72-320, Stanford university, CA, 1972.
[13] B. Baumgart, Winged-edge polyhedron representation for computer vision, National Computer Conference, 1975.
[14] L. Guibas, J. Stolfi, Primitives for the manipulation of general subdivisions and the computation of voronoi diagrams, ACM Trans. Graph. 4 (2) (1985) 74–123.
[15] H. Brahana, Systems of circuits on two-dimensional manifolds, Annals of Mathematics (1921) 144–168.
[16] P. Lienhardt, *N*-dimensional generalized combinatorial maps and cellular quasi-manifolds, International Journal on Computational Geometry and Applications 4 (3) (1994) 275–324.
[17] M. Worboys, Event-oriented approaches to geographic phenomena, International Journal of Geographic Information Science 19 (1) (2005) 1–28.
[18] L. Cardelli, P. Gardner, Processes in space, in: Computability in Europe, 2010, pp. 1–21.
[19] M. John, R. Ewald, A. Uhrmacher, A spatial extension to the π calculus, Electronic Notes in Theoretical Computer Science 194 (3) (2008) 133–148.
[20] J. Jiang, M. Worboys, Event-based topology for dynamic planar areal objects, International Journal of Geographical Information Science 23 (1) (2009) 33–60.
[21] L. Cardelli, Brane calculi, in: V. Danos, V. Schachter (Eds.), Computational Methods in Systems Biology, Springer, 2005, pp. 257–278.
[22] R. Barbuti, A. Maggiolo-Schettini, P. Milazzo, G. Pardini, L. Tesei, Spatial P systems, Natural Computing 10 (1) (2010) 3–16.
[23] T. Damgaard, E. Højsgaard, J. Krivine, Formal cellular machinery, in: V. Danos, V. Schachter (Eds.), Proc. SASB 2011, 2011.
[24] J. Krivine, R. Milner, A. Troin, Stochastic bigraphs, Electronic Notes in Theoretical Computer Science 218 (2008) 73–96.