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Commonsense notions of proximity and direction in environmental space

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It is desirable that formal theories of qualitative reasoning should be informed by the ways in which humans conceptualize the spaces in which they live. The work described in this paper uses data provided in experiments with human subjects to derive some regularities in such conceptualizations. The data concerns human conceptualization of proximity and direction within a university campus. The results are analyzed using several approaches. In particular, the relationship between geometric and human conceptual models of the space is explored; the structure and regularities of combinations of proximity and direction relations are examined; and the issue of granularity in vague spatial relations is considered. Overall, the results show that while individual differences between humans are important, there are striking regularities in the population's notions of distance and direction in the space. The paper concentrates primarily on the formal foundations of commonsense notions of proximity and direction, but also identifies links to more applied domains, such as mobile and location-aware navigation systems.

Keywords: Qualitative spatial reasoning, vagueness, granularity, near, left

1 Introduction

Qualitative spatial reasoning is concerned with representing and reasoning with our commonsense knowledge of the spatial aspects of the physical world (Cohn and Hazarika, 2001). There is a need for formal theories of spatial representation and reasoning to be properly guided by the way humans actually think about space, i.e., to be *cognitively informed*. Proximity and direction are two basic components of an ontology of space, and this paper describes experiments with human subjects on aspects of these categories. The spaces investigated are at the *environmental* scale, that is at the scale of "buildings, neighborhoods, and cities" (Montello, 1993). Such spaces cannot be apprehended in a single viewing, and useful knowledge of them can be gained only by a series of observations over time and from different locations in the space. This sets them apart from "table-top" spaces, where the space and its constituent entities can be more or less taken in at one observation (Zubin, 1989).

Human conceptualizations of the spatial aspects of the physical world are subject to a wide variety of distortions. For example, Sadalla et al. (1980) observed asymmetries in the human perception of nearness, with more significant *reference points* or *landmarks* generally being understood to be near to adjacent points more frequently than vice versa. Stevens and Coupe (1978) and Hirtle and Jonides (1985) provide evidence of distortions in human spatial cognition resulting from the apparent hierarchical arrangement of places according to spatial and semantic criteria. Distortions such as asymmetry, vagueness, landmarks, hierarchies, and clustering have proved difficult to integrate with the logical systems, which form the backbone of computational approaches to qualitative spatial reasoning. Nevertheless, Tversky (1992) argues that these distortions are important cognitive devices that help humans to organize spatial information. Ideally, cognitively informed computational models of qualitative spatial information should be able to allow for such distortions.

In this paper we address three specific issues related to a cognitively informed qualitative theory of vague spatial relations in environmental space:

1. the relationship between geometric and human conceptual models of an environmental space;
2. the structure and regularities of combinations of proximity and direction relations; and
3. the role of granularity in vague spatial relations.

Although this paper concentrates on the formal foundations of commonsense notions of proximity and direction, the three issues identified above have clear links to more applied domains. Consider a mobile location-aware navigation system designed to help users navigate around an unfamiliar environment, such as a university campus. Support for commonsense notions of space will help users to interact more efficiently, accurately, and intuitively with the navigation system. For example, translating between the crisp geometric route information stored in a spatial database and the vague spatial relations used in human-centered queries and navigation instructions requires an understanding of the relationship between geometric and human conceptual models of space. Similarly, the ability to infer new information from combinations of vague spatial relations is important in navigation systems, where neither direction nor proximity alone are sufficient to provide clear navigation instructions.

After reviewing relevant literature in section 2, section 3 sets out the experimental design used in this work. Two experiments with human subjects concerning the vague spatial relations "near" and "left" in environmental space are described. Some results on the first experiment on nearness have been reported in Worboys (2001) and Duckham and Worboys (2001). Since the publications of those papers, the work has moved on, and in this paper we summarize the results from earlier work, describe results of the direction experiment, and discuss some of the population-level characteristics of both. Section 4 introduces the concept of *support metrics* to summarize the results of the experiment. Section 5 explores the relationship between the geometry of the campus space and the human conceptualization of that space inferred from the experimental results. Section 6 provides a formal structure for the human conceptualization of the space, uncovering some of the structure concerned with the integration and granularity of proximity and direction relations. Section 7 concludes the paper with a summary and outlook for future work.

It is important to say something about the limitations of the empirical work. The experiments are limited by the small size of the cohort and the similarity of backgrounds of its members, and relate to a single spatial domain. However, the the results of this work do show that while individual differences are important, there are striking regularities in the population's notions of distance and direction in the space. It should also be noted that while the work does focus on qualitative linguistic terms, such as "near" and "left," it does not address the more general question about the degree to which human spatial knowledge is qualitative in general.

2 Literature background

Several disciplines in cognitive science have investigated the semantics of spatial relations. Fundamental work analyzing spatial concepts has been conducted, for example, in linguistics, psycholinguistics, and psychology (Clark, 1973; Miller and Johnson-Laird, 1976; Talmy, 1983; Herskovits, 1986; Vandeloise, 1991; Taylor and Tversky, 1992a,b; Levinson, 1996; Taylor and Tversky, 1996). Besides topological relations, two important classes of spatial relations are direction and metric relations. Freeman (1975) described 13 spatial relations between objects, including “to the left of” and “near.” Peuquet (1986) reduced Freeman’s classification to two basic spatial concepts, direction and distance.

A detailed overview of the work on nearness and proximity relations has already been given in Worboys (2001); our overview focuses on direction relations, in particular on leftness relations. Leftness relations are a special case of projective relations. Projective relations, and the frames of reference on which they depend, are discussed in section 2.1. Section 2.2 discusses qualitative approaches to describing both proximity and direction relations.

2.1 Projective relations and frames of reference

Spatial relations described by *projective terms* (Herskovits, 1986), such as “to the left of” or “in front of,” are called *projective relations*. Approaches to projective relations are typically axis- or region-based approaches. Tversky (1996) provides an interpretation of projective terms using an underlying spatial framework that consists of body axes. Herskovits (1985), however, suggests that the location specified by a projective term is either in a half plane, in a sector, or a position that is on or close to an axis. Logan and Sadler (1996) propose *spatial templates* as regions of acceptability. A spatial template distinguishes three regions in terms of acceptability: given a spatial relation there are locations that are good, acceptable, and unacceptable instances. The regions of acceptability are not crisp but gradually blend into each other. Hayward and Tarr (1995) found that in both spatial language and visual perception there are prototypical effects that lead to similarities in the preferred locations for objects connected by a spatial relation. The selection of a spatial term like “to the left of” or “above” depends on angular information (Zimmer et al., 1998). Distance- and coordinate-based approaches to prepositions describing spatial relations are given by Miller and Johnson-Laird (1976) and Garnham (1989), respectively.

The use of projective relations depends on an underlying frame of reference (Levelt, 1984; Retz-Schmidt, 1988; Levelt, 1996; Levinson, 1996). In a projective relation between two objects, one object is the *reference object* to which the second object, called the *located object*, is related. In linguistics the corresponding notions are termed *ground* and *figure*, respectively (Talmy, 1983). Levinson (1996) distinguishes three frames of reference: a *relative* (or *deictic*), an *intrinsic*, and an *absolute* frame of reference. For a relative frame of reference the projective relation depends on the orientation and location of an observer and the locations of the reference and located objects. For an intrinsic frame of reference the projective relation is determined by the orientation of the reference object itself and the locations of the reference and located objects. For an absolute frame of reference (like that used for cardinal directions) the spatial relation is determined by the locations of the reference and located objects alone.

A related terminology for describing frames of reference has been introduced by Hart and Moore (1973): *egocentric* and *allocentric* frames of reference (see also Klatzky, 1998). In the egocentric case the position (and orientation) of an observer is required for the description of a projective relation. In the allocentric case the relation depends only on the reference object, i.e. is independent of an observer. In the terminology of Levinson (1996), "egocentric" is approximately equivalent to "relative," while "allocentric" encompasses both "intrinsic" and "absolute" frames of reference.

2.2 Qualitative approaches

Several attempts to formalize various aspects of spatial relations can be found in the qualitative spatial reasoning literature (for an overview see Cohn and Hazarika, 2001). In addition to inherently qualitative work on topological relationships (Egenhofer and Franzosa, 1991; Randell et al., 1992) and shape (Cohn, 1995; Clementini and Di Felice, 1997; Galton and Meathrel, 1999), there has also been some work on computational aspects of qualitative distance (Hernández et al., 1995; Gahegan, 1995), qualitative direction (Ligozat, 1993), qualitative reasoning with cardinal directions (Frank, 1996), and qualitative positional information (Clementini et al., 1997). Spatial reasoning about distances and directions is investigated by Frank (1992), the integration of topology and direction is proposed by Sharma and Flewelling (1995). Approaches using ordering information to describe direction-based relations are given by Hernández (1992), Freksa (1992), Schlieder (1995), and Eschenbach and Kulik (1997). Hernández (1992) introduces a sector-based model that assumes a local neighborhood for each object that is partitioned into eight sectors: "left," "left-front," "front," and so forth. Freksa (1992) and Schlieder (1995) formalize relations using three points. Freksa's model uses three axes and six regions, generated by three points, whereas Schlieder's model is based on triangle orientations. Eschenbach and Kulik (1997) present an axiomatic description of projective relations that is based on the geometric concepts of incidence and betweenness.

Indeterminacy in spatial relations and spatial locations has also been addressed in the literature. Freeman (1975) suggests modeling indeterminate spatial relations using fuzzy relations. Similarly, Herskovits (1986) argues that "to the right" can be seen as a graded concept that can be applied in different degrees. As mentioned above, Logan and Sadler (1996) share this graded view of spatial relations, providing acceptability regions that are not sharply bounded. The "egg-yolk" calculus (Cohn and Gotts, 1996) has proved useful as a qualitative framework for reasoning about indeterminate regions. Robinson (1990, 2000) used an adaptive algorithm to produce a fuzzy membership function for nearness, based on experiments with human subjects.

Although there is a long tradition in linguistics and psychology of investigating spatial relations, there is much less work providing computational models of spatial relations, in particular for "nearness" and "leftness" relations. Some research has begun to provide a basis for closer integration between the computational and the cognitive. Kuipers (1977, 1978) suggested the *Tour* model to represent a person's knowledge of its environment. Kuipers assumes that our spatial knowledge is organized as a *cognitive map*, and tailored the Tour model to cope with incomplete knowledge and to answer route-finding and relative-position problems. Chown et al. (1995) proposed a general, integrated model (PLAN) of spatial knowledge in large-scale space. Gapp (1994) suggested a computational model for spatial relations and evaluated the applicability of projective relations with respect to angular, distance, as well as shape information in Gapp (1995). Our experiments and computational model deal with nearness and projective relations for objects in environmental space.

3 The experiments

Data was collected from human subjects in two structurally similar experiments, concerned with nearness and leftness, respectively. The experiments concerned places on the campus of Keele University, UK (see Figure 1), an area of 600 acres set within landscaped grounds. The area is well suited for the experiments, being neither abnormally flat nor hilly; having an irregular network of paths, buildings, and roads; having no highly dominant landmarks; and being of a size to be classified as an environmental space.

The experiments are designed to elicit levels of support for spatial relationships between places in the space. As with many qualitative spatial relationships, nearness and leftness are vague, in the sense that determination of truth or falsity of the relationships may be unclear for some borderline cases. The form of the experiments, derived from ideas in Bonini et al. (1999), is designed to determine the degree of support indirectly. We have also factored out individual differences of place, so that lines of visibility, obstacles, landmarks, and paths are not considered in this work, except in a generalized sense. The experiments are qualitative in nature, exploring subject's views of "nearness" and "leftness" relationships between assemblages of places in the space. However, the analyses in later sections are often quantitative, making use of statistical techniques for significance testing and Dempster-Shafer theory for analysis of degrees of belief.

For the nearness experiment, a group of 22 human subjects were asked to complete a series of questionnaires concerning the nearness of places on the campus to each other. Just over twenty places were selected as being well known ("significant") places on campus, identified using a preliminary study. Half the subjects (the *truth group*) were asked in a questionnaire to check on the list of significant places those places for which it was true to say that they were near to a fixed *reference place*, drawn from the collection of significant places. The other half of the

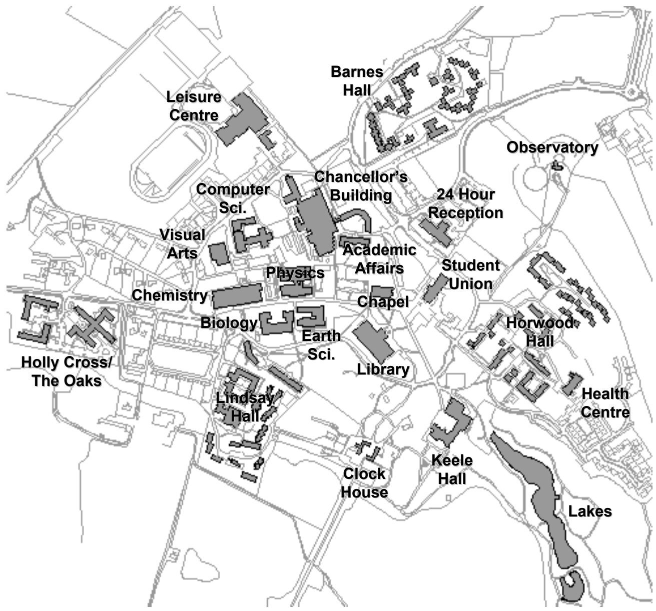

Figure 1: Significant places on the Keele campus

subjects (the *falsity group*) were asked to identify those places for which it was false to say that they were near to the same reference place. With a break of at least one day between successive questionnaires, each subject was then asked to complete further questionnaires, one for each reference place, until information about all reference places had been gathered for each subject. All the subjects were Keele University staff with some years experience of the campus and were asked to complete questionnaires without thinking too much and certainly without reference to maps. Full details of the nearness experiment with some analysis of results is contained in Worboys (2001) and Duckham and Worboys (2001).

The leftness experiment was of the same form. In this case the Library was taken as fixed throughout the questionnaires. In each questionnaire, subjects were asked to imagine standing at one of the reference places facing the Library, and to identify each place for which it was true (false) to say that that place was on the left. As with the nearness experiment, each subject was asked to complete questionnaires, one for each reference place, until information about all reference places had been gathered for each subject. Vague directional predicates such as “left” and “right” are commonly found in the literature on spatial cognition and so were obvious choices for this study. Leftness is complementary to nearness, and so supports our aim of providing a basis for integrating vague spatial relations. It was assumed that there would be no substantial differences between a study of leftness or rightness (aside from the reversal of direction), and so the decision to study “left” rather than “right” was an arbitrary one.

For each reference place and each spatial relation, tallies were made of which places were checked by the two groups. For example, Table 1 shows the tallies for leftness where the reference place is the Chancellor’s Building. (Throughout this paper, leftness is always taken to be with respect to the Library).

Place	Truth Group	Falsity Group	Place	Truth Group	Falsity Group
24 hour Reception	9	0	Holly Cross	0	11
Academic Affairs	5	3	Horwood Hall	9	2
Barnes Hall	8	3	Keele Hall	5	5
Biological Sciences	0	10	Lakes	7	4
Chancellors Building	0	11	Leisure Centre	2	6
Chapel	6	2	Lindsay Hall	0	11
Chemistry	0	11	Observatory	8	0
Clock House	2	7	Physics	0	11
Computer Science	0	11	Student Union	9	1
Earth Sciences	0	10	Visual Arts	0	11
Health Centre	7	2			

Table 1: Leftness tallies for the Chancellor’s Building

One important difference in the phrasing of the nearness and leftness questions needs to be highlighted. In the terminology of Hart and Moore (1973), introduced in section 2, the nearness experiment used an allocentric question, concerning the nearness of places independently of the respondent’s location or perspective. The leftness experiment used an egocentric question, which explicitly asked respondents to imagine standing at a particular location facing in a particular direction. The reason for this change was that preliminary investigations suggested that allocentric forms of the leftness question were generally harder to understand than more natural egocentric versions. Because not all buildings possess an intrinsic frame of reference (a front and a back, for example), allocentric questions about leftness are difficult to phrase clearly without introducing additional complexities, such as asking respondents to imagine vectors connecting different reference places in the study. Consequently, the egocentric form of the leftness question was used, since ease of comprehension had to take precedence over other concerns. In fact, some authors have argued that any spatial cognitive task utilizing working memory will be organized using an egocentric perspective (Montello, 1992). As a result, the choice of egocentric or allocentric question may not have a significant effect upon how the spatial task is completed.

In both experiments, each particular questionnaire contained a reference place and a list of all other significant location to be related to the reference place. Apart from the task itself, this list provided the only context for the subjects.

4 Results and support metrics

Let P denote the set of significant campus places and let Q denote the set of significant campus places except for the Library, $Q = P \backslash \{\text{Library}\}$. For places $p, q \in P$, let the binary relation $n(p, q)$ be taken to indicate that p is near to q. For places $p, q \in Q$, $l(p, q)$ indicates that p is to the left from the point of view of a person positioned at q facing the Library. The experiments provide collections of data supporting the truth and falsity of $n(p, q)$ and $l(p, q)$, for all significant campus places. This data is in the form of a tally of votes for the truth or falsity of the corresponding statements about $n(p, q)$ and $l(p, q)$. The tallies are contained in four integer arrays:

$$\begin{array}{l} \{n_\top(p,q)|p,q \in P\} \\ \{n_\bot(p,q)|p,q \in P\} \\ \{l_\top(p,q)|p,q \in Q\} \\ \{l_\bot(p,q)|p,q \in Q\} \end{array}$$

where $n_\top$ and $n_\bot$ indicate the number of subjects voting for the truth or falsity, respectively, of the nearness relation and $l_\top$ and $l_\bot$ have a similar meaning for the leftness relation.

Let the total number in each of the truth and falsity groups be N. Support for the truth or falsity of $n(p, q)$ and $l(p, q)$ is indicated by the tallies in the arrays $n_\top$, $n_\bot$, $l_\top$, and $l_\bot$. The issue addressed here is how to measure this support. The most simple-minded approach is to take the difference between the tallies for the truth and falsity groups to generate support metrics σ_ν, σ_λ as follows:

$$\sigma_\nu(p,q) = \frac{n_\top(p,q) - n_\bot(p,q)}{N} \tag{1}$$

$$\sigma_\lambda(p,q) = \frac{l_\top(p,q) - l_\bot(p,q)}{N} \tag{2}$$

Metrics σ_ν and σ_λ lie between -1 and 1, where a value of -1 indicates complete support for the falsity of the relation, zero indicates no support either way, and 1 indicates complete support for its truth. Another approach is to use statistical tests to determine the significance of support for or against the relations $\nu(p, q)$ and $\lambda(p, q)$. Examples of the latter approach can be found in previous analyses of the nearness data (Worboys, 2001; Duckham and Worboys, 2001).

However, both these approaches, as well as approaches based upon fuzzy logic, conflate evidence for and against the propositions. An alternative is to use Dempster's rule of combination (Shafer, 1976) to give support metrics. This yields the following metrics σ_ν^+ and σ_λ^+ for degree of support for the truth of the nearness and leftness relationships, respectively.

$$\sigma_\nu^+(p,q) = \frac{n_\top(p,q)(N - n_\perp(p,q))}{N^2 - n_\top(p,q)n_\perp(p,q)} \tag{3}$$

$$\sigma_\lambda^+(p,q) = \frac{l_\top(p,q)(N - l_\perp(p,q))}{N^2 - l_\top(p,q)l_\perp(p,q)} \tag{4}$$

The metrics σ_ν^- and σ_λ^- for degree of support for the falsity of the nearness and leftness relationships, respectively, are given by:

$$\sigma_\nu^-(p,q) = \frac{n_\perp(p,q)(N - n_\top(p,q))}{N^2 - n_\perp(p,q)n_\top(p,q)} \tag{5}$$

$$\sigma_\lambda^-(p,q) = \frac{l_\perp(p,q)(N - l_\top(p,q))}{N^2 - l_\perp(p,q)l_\top(p,q)} \tag{6}$$

All support functions take values that lie between 0 and 1, where 0 indicates no support and 1 indicates full support for the appropriate proposition.

5 Commonsense geometry and the surveyed world

An interesting question concerns the relationship between people's commonsense views of nearness and leftness and the "objective" world measured by surveyors and mapped by cartographers. In order to understand overall effects it is important to "normalize" the data. In the case of the nearness experiment, each questionnaire provides very limited context to the respondent; a reference place, list of target places, and a question about the truth/falsity of the nearness relation between the reference place and each of the targets. However, the list of places does provide some context, and an initial question is whether this context influences the response. To see that it does have a partial effect, compare Figures 2 and 3.

In Figure 2, support $\sigma_\nu(p,q)$ is plotted against distance on the ground, $|p-q|$, and one sees an expected relationship where as distance increases so support for nearness decreases. In Figure 3, the distance has been normalized to take account of the contexts provided by the lists of places, so for reference places on the periphery of the campus, where distances to targets is on average further than for central reference places, a compensating factor (based on the square root of the mean of squares of distances to the reference) has been applied. Figure 3 shows the "sharpening up" of the resulting distribution. The graph of the Dempster support function σ_ν^+ plotted against the normalized distances is shown in Figure 4.

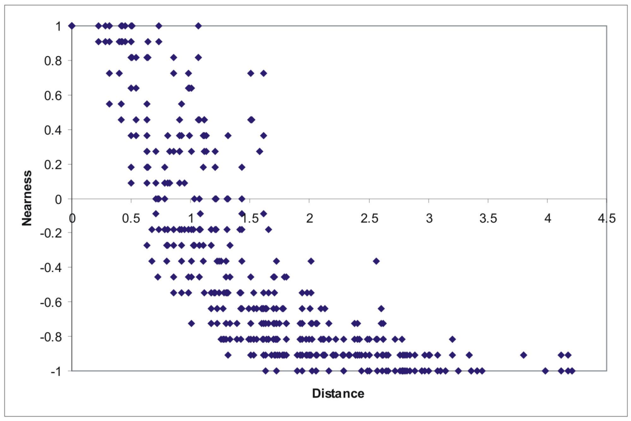

Figure 2: Variation of $\sigma_\nu(p,q)$ with distance

To see the overall structure of the leftness data, the positions of places have been normalized as follows. For places $p, q \in P$, the data pertains to the situation where a respondent is asked to evaluate truth/falsity of leftness of p from the viewpoint of being positioned at q and facing the Library. A coordinate frame is set up with the Library fixed in position (1,0), and q at the origin. The *normalized position* of p is then given by its coordinates in this coordinate frame. Figure 5 shows the result of plotting support $\sigma_\lambda(p,q)$ against the angular component (in degrees) of the normalized position of p for all $p, q \in P$.

Figure 5 strikingly shows the regularities in correspondence between human perception of spatial relationships, and those relationships as surveyed on the ground and presented in conventional maps. The overall shape of the "curve" is not unlike the familiar sine wave for the trigonometry of Euclidean space. It may be noticed that there are some significant errors in perception of angular relations in a few cases. Three places at an angle of between $120°$ and $135°$ have been voted not left. This reflects people's perceptions of positions on the campus. Special cases such as this need further investigation for topographic interferences such as curved roads, hills, and landmarks.

Two further features of the graph may be pointed out. The first is the slight lack of rotational symmetry about the origin. The intercept with the horizontal axis occurs at between about $0°$ and $30°$. This reflects the slight lack of symmetry in the leftness relation. If something is directly ahead of us, it is not just that we are undecided about its leftness, but we are likely to consider it not to our left. The second feature is the maximum of the curve occurs at an angle less than $90°$. This indicates a perhaps more surprising feature of human understanding of the leftness relation, where we feel most strongly that something is to the left

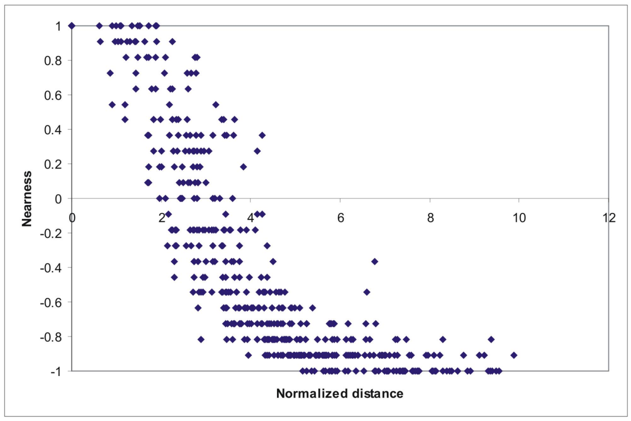

Figure 3: Variation of $\sigma_\nu(p, q)$ with normalized distance

of us if it is almost perpendicular to the direction of view *and slightly in front*. An alternative interpretation of the second feature is that the curve's maximum appears flat from $60°$ to $100°$, suggesting that due $90°$ is maximally left, but so is a range that is a little biased toward the front half of the space. A little forward of due left is more likely to be called 'left' than a little backward of due left. This property, and a corresponding property for the minimum, deserve further analysis.

Another visualization of these features is shown in Figures 6 and 7, where Dempster's support metrics σ_λ^+ and σ_λ^- are plotted against the same normalized angle. The advantage of this visualization is that the two separate support metrics, measuring evidence for and against the leftness relation, enable some further insight. As can be seen, supporting evidence for leftness increases sharply around $0°$, but is not zero at $0°$. However, evidence against leftness at $0°$ is substantial at around the 0.6 level. So, it is not that there is no commitment either way about leftness at $0°$, but that there is commitment both for and against, with the weight of evidence very much against. We can also see from the graphs that there is some support for leftness in directions behind the viewer at angles in the approximate range $-180°$ to $-135°$ (that is, in directions to the right and behind of the viewer). This phenomena needs further analysis, but could be because humans have more difficulty determining directions for places behind their mind's eye field of view.

6 Properties of three-valued proximity and direction relations

We may further analyze the structure and interactions between nearness and leftness relations using a three-valued interpretation of the data. In this section we confine our analysis to Dempster's metrics for the degree of support for the truth

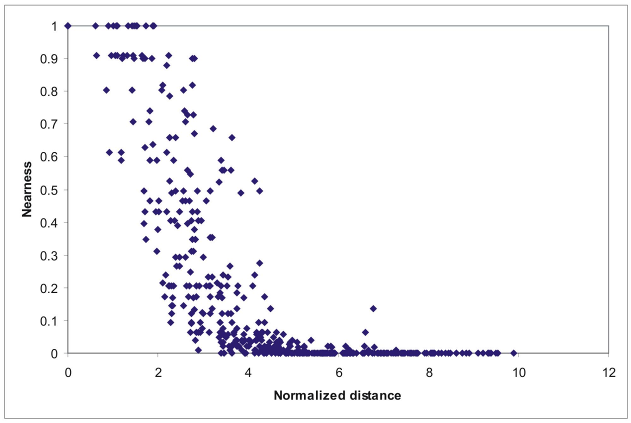

Figure 4: Variation of $\sigma_\nu^+(p,q)$ with normalized distance

of the nearness and leftness relations, σ_ν^+ and σ_λ^+, respectively, although a similar analysis could also be performed for σ_ν^- and σ_λ^-.

6.1 Nearness properties

We first define two threshold values, h_ν and l_ν, such that $0.0 \leq l_\nu \leq h_\nu \leq 1.0$. We return to the question of how to determine the value of these thresholds later in this section; for now we simply assume they are known. Using these thresholds we can define a three-valued nearness relation ν as follows:

$$\forall p,q \in P \quad \nu(p,q) = \begin{cases} \top & \text{if } \sigma_\nu^+(p,q) > h_\nu \\ \bot & \text{if } \sigma_\nu^+(p,q) < l_\nu \\ ? & \text{otherwise} \end{cases} \tag{7}$$

The intuition behind these definitions is that the statement $\nu(p,q) = \top$ indicates that it is definitely the case that p is near q. Conversely, $\nu(p,q) = \bot$ indicates that it is definitely not the case that p is near q. Otherwise, it is indeterminate whether p is near q ($\nu(p,q) = ?$).

The notion of "nearness" is a *similarity* relation, where the formal properties of equivalence are weakened. Nearness may be assumed to be reflexive by definition, but symmetry is not born out by this work. For example, for a wide range of threshold values, the Chapel may be judged to be definitely near to Academic Affairs, ν('Chapel', 'Academic Affairs') $= \top$, but the nearness of Academic Affairs to the Chapel is indeterminate, ν('Academic Affairs', 'Chapel') $=?$. Using the three truth values, the results do exhibit *weak symmetry*, defined below.

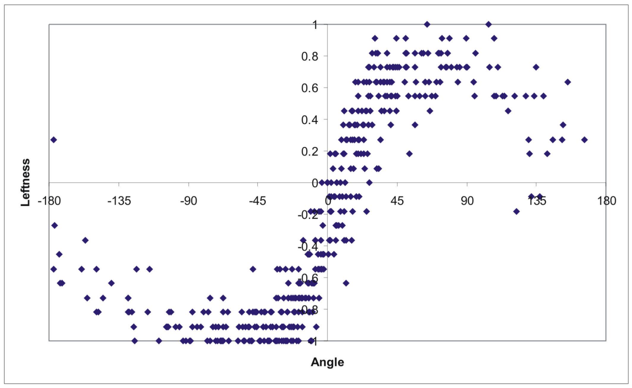

Figure 5: Variation of $\sigma_\lambda(p,q)$ with angle

$$\forall p,q \in P \quad \nu(p,q) = \top \rightarrow \nu(q,p) \neq \bot \tag{8}$$

While as expected the nearness relation is not transitive, it does also exhibit *weak transitivity*:

$$\forall p,q \in P \quad p\nu q = \top \wedge q\nu r = \top \rightarrow p\nu r \neq \bot \tag{9}$$

As mentioned above, we discuss the choice of values for thresholds h_ν and l_ν later, in section 6.5. At this point, we simply note that for a wide range of threshold values the properties of weak symmetry and weak transitivity hold.

6.2 Leftness properties

Using further thresholds h_λ and l_λ where $0.0 \leq l_\lambda \leq h_\lambda \leq 1.0$, we can define a three-valued leftness relation in a similar way to the nearness relation.

$$\forall p,q \in Q \quad \lambda(p,q) = \begin{cases} \top & \text{if } \sigma^+_\lambda(p,q) > h_\lambda \\ \bot & \text{if } \sigma^+_\lambda(p,q) < l_\lambda \\ ? & \text{otherwise} \end{cases} \tag{10}$$

Leftness is assumed to be irreflexive by definition, $\lambda(p,p) = \bot$. The data shows that the leftness relation is antisymmetric:

$$\forall p,q \in Q \quad \lambda(p,q) = \top \rightarrow \lambda(q,p) = \bot \tag{11}$$

The leftness relation also exhibits a weak property related to transitivity.

$$\forall p,q \in Q \quad \lambda(p,q) = \top \wedge \lambda(q,r) = \top \rightarrow \lambda(r,p) \neq \top \tag{12}$$

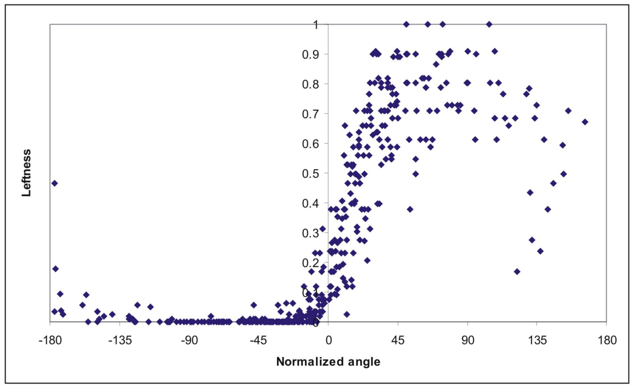

Figure 6: Variation of $\sigma_{\lambda}^{+}(p, q)$ with angle

Again, both these properties hold for a wide range of threshold values, discussed in more detail in section 6.5.

6.3 Properties combining nearness and leftness

Data exploration and visualization techniques were developed to help reveal a family of properties that relate nearness to leftness, allowing stronger inferences than can be achieved by using either relation in isolation. Such properties address the need to combine evidence from several observations of nearness and leftness so as to constrain the possible spatial relationships that can then be obtained. Our analysis focused on inferences involving nearness and leftness relations between three places. Again, we note that all the properties in this section hold across a range of threshold values. However, exhaustively searching for such inferences was impossible without knowing in advance what threshold values to use.

The approach was as follows. For all $p, q, r \in Q$, we related the 81 possible values of $\nu(p, q)$, $\nu(q, p)$, $\lambda(p, q)$, and $\lambda(q, p)$ (each can take one of three values $\top$, ?, $\bot$) to the 81 possible values of $\nu(q, r)$, $\nu(r, q)$, $\lambda(q, r)$, and $\lambda(r, q)$. The number of distinct conclusions in the data for each pair of premises was then plotted graphically on the map (red for one distinct conclusion, orange for two, yellow for three or more). Finally, by interactively varying the threshold values it was possible to gain an impression of the inferences that were stable across a range of thresholds (those that remained red), and so warranted further investigation.

A typical property that relates nearness and leftness based on three places p, q, and r, initially identified using the analysis technique above, can be stated as follows:

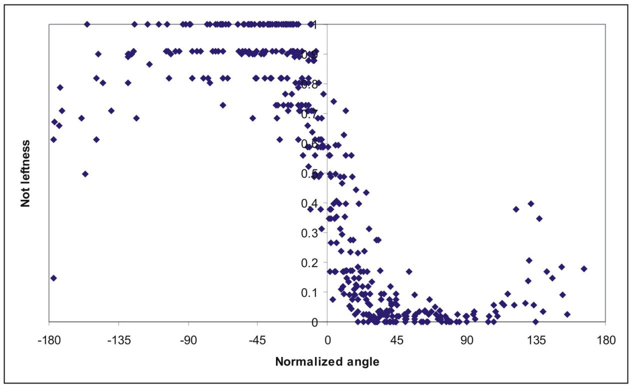

Figure 7: Variation of $\sigma_{\lambda}^{-}(p,q)$ with angle

$$\forall p,q \in Q \quad \nu(p,q) = \bot \wedge \nu(q,r) = \top \wedge \lambda(p,q) = \top \rightarrow \lambda(r,p) = \bot \wedge \lambda(p,r) \neq \bot \quad (13)$$

In order to help the reader gain some intuition relating the property in equation 13, the diagram in Figure 8 is provided. Figure 8 includes the location of the Library for reference, but it is important to realize that the property in Figure 8 holds for all significant places $p, q, r \in Q$ independently of any direct reference to the Library.

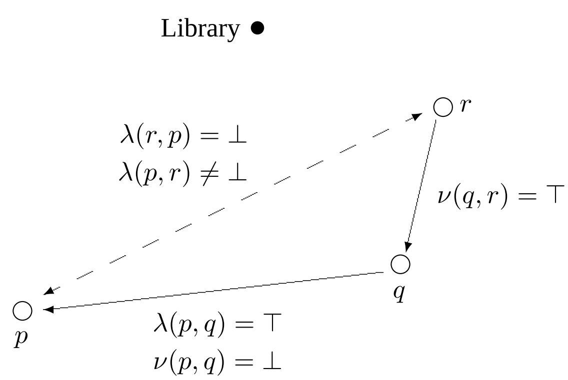

Figure 8: Diagram illustrating property in equation 13

An intriguing feature of the data is that similar properties hold even when the direction of either or both of the nearness relations is reversed. More formally, the following more general property holds for a wide range of threshold values:

$$\forall p, q \in Q \quad (\nu(p,q) = \bot \vee \nu(q,p) = \bot) \wedge (\nu(q,r) = \top \vee \nu(r,q) = \top) \\ \wedge\, \lambda(p,q) = \top \rightarrow \lambda(r,p) = \bot \wedge \lambda(p,r) \neq \bot \quad (14)$$

Further, the converse property where the direction of the leftness relation $\lambda(p,q)$ is reversed also holds for a wide range of threshold values irrespective of the direction of the nearness relations.

$$\forall p, q \in Q \quad (\nu(p,q) = \bot \vee \nu(q,p) = \bot) \wedge (\nu(q,r) = \top \vee \nu(r,q) = \top) \\ \wedge\, \lambda(q,p) = \top \rightarrow \lambda(p,r) = \bot \wedge \lambda(r,p) \neq \bot \quad (15)$$

Together, these properties summarize some of the basic structure of the data in a way that could be used for a qualitative reasoning system. A cognitively informed model of proximity and direction in an environmental space should take account of these properties.

6.4 Granularity and qualitative vector spaces

This section discusses and formalizes the granularity imposed on the space by imperfect knowledge resulting from qualitative notions of nearness and leftness. In terms of navigation, it addresses questions like "If my observations tell me that I am near to place X and to the left of place Y, what is the range of possible places at which I may be located?"

At any place on the campus (not necessarily one of the reference places), there will be nearness and leftness relations to each of the reference places. If we take the three-valued relations, then with each place we can associate a vector of values from the set $\{\top, ?, \bot\}$, where the components of the vector give the values of the nearness and leftness relations of the location of the person to each of the reference places. We can define a relation on the set of campus locations, where two places are related if they have the same vector of values, and it is clear that this relation is an equivalence relation. As there are only a finite number of such vectors, the campus may be partitioned into a finite number of blocks. This equivalence relation can be thought of as an *indiscernibility* relation, in that places with the same vectors are indiscernible in terms of their relationships to the reference places. In this way, a *granularity* is imposed on our campus space (cf. Hobbs, 1985). If we reduce the number of reference places, or the set of leftness and nearness relationships under consideration, then the blocks of the partition will be larger, until in the limiting case where there are no reference places to relate to, there is a single block: the whole campus. This section explores the relationship between reference relationships and granularity, using the notion of a *qualitative vector space* (QVS), introduced in Duckham and Worboys (2001) as part of a previous analysis of the nearness relation.

Formally, a QVS is constructed from a subset of three-valued relations for an environmental space (the campus in this study), termed a *framework*. For any place $x \in Q$ we define the predicates $\nu_p(x) = \nu(x, p)$ and $\lambda_q(x) = \lambda(x, q)$ for some $p \in P$ and $q \in Q$. Next, define the set G to be the union of all nearness and leftness predicates, $G = \{\nu_p | p \in P\} \cup \{\lambda_q | q \in Q\}$. A framework F is then a subset of G, $F \subseteq G$.

For example, nearness to the Clock House and leftness of the Chancellor's Building forms a framework, $F' = \{\nu_{\text{Clock House}}, \lambda_{\text{Chancellor's Building}}\}$. It is then possible to describe the position of every other place in the space with respect to this framework. Table 2 provides a partial list of nearness and leftness values for this framework F'.

Place	$\nu_{\text{Clock House}}$	$\lambda_{\text{Chancellor's Building}}$
24 hour Reception	$\bot$	$\top$
Academic Affairs	$\bot$	?
Barnes Hall	$\bot$	?
Biological Sciences	?	$\bot$
Chancellor's Building	$\bot$	$\bot$
Chapel	$\bot$	?
...	...	...

Table 2: Example framework $F' = \{\nu_{\text{Clock House}}, \lambda_{\text{Chancellor's Building}}\}$

The partition of the space induced by F' can be visualized as shown in Figure 9. In Figure 9, regions have been drawn on the map of the campus, where each region contains all those places that share the same qualitative vector. For example, the experimental data indicates that the 24 Hour Reception and the Observatory are both considered to be definitely not near to the Clock House and definitely left from the Chancellor's Building (when facing the Library). Consequently, both places have the qualitative vector $(\top, \bot)$ with respect to the framework F'. No other places are described by this qualitative vector, so the region labeled $(\top, \bot)$ contains only these two places, the 24 Hour Reception and the Observatory. We say that the 24 Hour Reception and the Observatory are *indiscernible* from one another with respect to the framework F'.

An important property of a framework is *faithfulness*, where the framework allows all places in the environmental space to be discerned. In the example of Table 2 and Figure 9 above, the framework F' is not faithful, since several different places share the same qualitative vectors (those that are indiscernible from one another with respect to the framework F'). However, adding two further predicates to the framework in Table 2, nearness to Physics and nearness to the Chancellor's

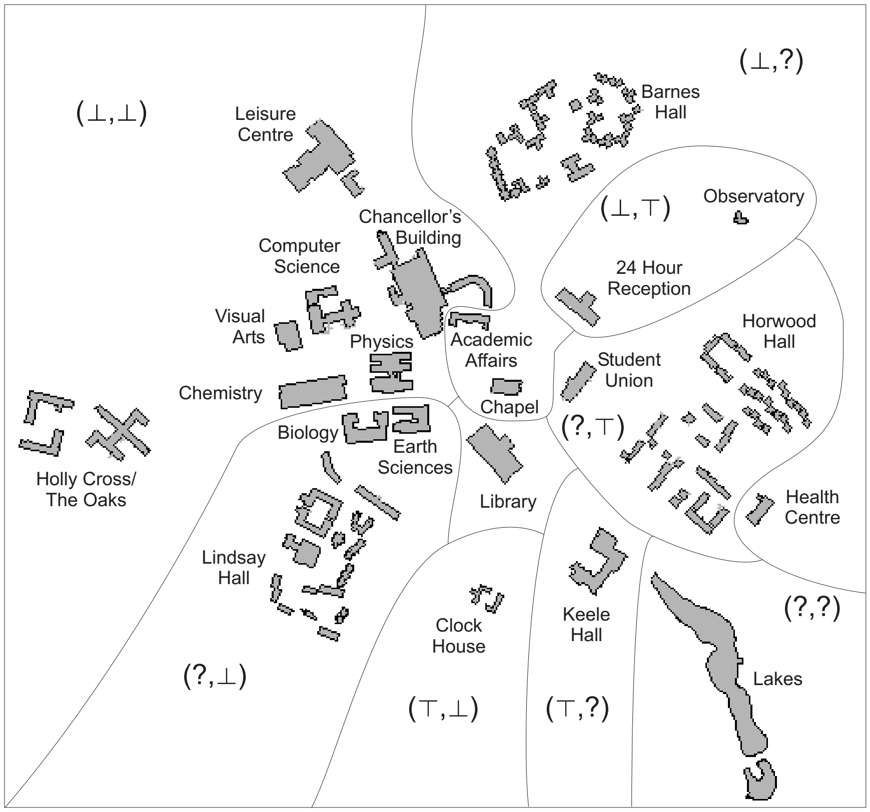

Figure 9: Partition induced by framework $F' = \{\nu_{\text{Clock House}}, \lambda_{\text{Chancellor's Building}}\}$

Building, yields a faithful framework. The significance of this result is that everywhere within our study space can be uniquely identified with reference to just four predicates: $\nu_{\text{Chancellor's Building}}$, $\lambda_{\text{Chancellor's Building}}$, $\nu_{\text{Clock House}}$, and ν_{Physics}. The theoretical minimum for the number of three-valued predicates needed to uniquely describe 21 places is $\log_3(21) \approx 3$, so this structure of four predicates represents a relatively efficient framework for the space.

A further property of faithful frameworks is *minimality*. The framework of four predicates described above is minimal in the sense that removing any one of the four predicates leads to a framework that is not faithful. In fact, this framework is the smallest minimal framework that could be found in the data set. An exhaustive search of all the frameworks in the data set was intractable assuming a computational complexity for the search procedure of $O(2^n)$. Consequently, a heuristic search of the data set was used, based on the well-known ID3 algorithm (see, for example, Munakata, 1998, Russell and Norvig, 2002). Given a faithful but not minimal framework, the heuristic preferentially discards the predicates associated with the least *information content*, determined in the usual way after Shannon and Weaver (1949). Thus, the heuristic cannot guarantee to find the smallest minimal framework, but previous studies have shown the technique to be an effective mechanism of finding near optimal solutions (Duckham and Worboys, 2001).

A similar search of nearness predicates alone yielded a smallest faithful minimal framework containing six predicates. With reference to the leftness predicates alone, there exists no faithful framework. This indicates that using both nearness and leftness together can result in a more efficient description of the study space than using either in isolation.

6.5 Threshold values

The question still remains as to how to assign values to the thresholds h_ν, l_ν, h_λ, l_λ. Each of the properties described in this section will hold only within a certain range of thresholds. In general, most of the properties are highly tolerant to variable thresholds, indicating that the properties are relatively stable. For example, Figure 10 shows the range of thresholds h_λ, l_λ for which all properties related to leftness hold (i.e. equations 11, 12, 14, 15). In Figure 10 the lower threshold l_λ corresponds to the abscissa and the upper threshold l_λ corresponds to the ordinate.

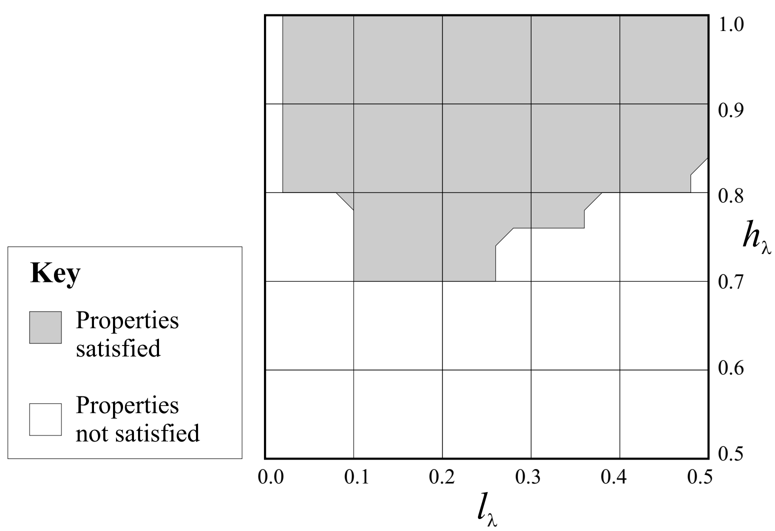

Figure 10: Solution space for thresholds h_λ and l_λ satisfying properties in equations equations 11, 12, 14, 15

The figure clearly shows the wide range of thresholds for which all the aforementioned properties hold (shaded gray region). In this example, the upper threshold, h_λ, seems to dominate the determination as to whether the properties hold. At h_λ values below 0.7 no value of l_λ leads to the properties holding. Conversely at h_λ values above 0.8 almost all values of l_λ lead to the properties holding. A similar diagram exists for the nearness thresholds, h_ν and l_ν (in relation to the properties in equations 8, 9, 14, and 15). Indeed, except for the difficulty of visualizing four dimensional spaces, we might also want to consider the solution space for all four thresholds simultaneously (in relation to equations 14 and 15).

In general, the size of smallest faithful minimal frameworks is more sensitive to threshold values than the properties in sections 6.1–6.3. However, it is still possible to find a set of threshold values for which all the properties described above are satisfied, in addition to yielding a smallest faithful minimal framework with just four relations. For example, the thresholds $h_\nu = 0.92$, $l_\nu = 0.10$, $h_\lambda = 0.72$, $l_\lambda = 0.26$ were found to provide all the desired properties and the smallest faithful minimal framework (four predicates). To find these thresholds we used the analysis outlined in section 6.3 to find interesting properties that hold at some threshold, and then determined at what threshold values all these discovered properties are satisfied.

7 Conclusions

This paper has reported results from two experiments with human subjects concerning conceptions of direction and angle in environmental space. The paper has demonstrated that although there are differences between individual subjects' views, and results vary from place to place depending on the relevant topography, there are striking regularities. The structure of individual differences has not been the subject of analysis. Lack of agreement between subjects could just reflect within-subject vagueness about whether a particular building is actually near/left or not, and the level of uncertainty gets turned into a binary decision differently by different subjects. Our summary comments and suggestions for further work are organized according to the three core issues enumerated in section 1: the relationship between commonsense and geometric notions of proximity and direction; combining proximity and direction relations; and granularity issues.

7.1 Commonsense and geometry

This paper has revealed a variety of regularities in the relationship between geometric and commonsense notions of proximity and direction. There exists a context-dependent relationship between commonsense notions of proximity and geometric distance. Commonsense notions of direction also exhibit striking similarities with their geometric counterparts. Using Dempster-Shafer support metrics avoids conflating evidence for and against a particular proposition, allowing situations where subjects tend to strongly disagree to be distinguished from those where subjects tend to offer no strong opinions either way.

The approach used in this paper is not limited to physical spaces, and might also be applied to virtual spaces, such as the WWW. Initial research applying the approach to a conceptual space, based on philosophical concept of “moral distance” has already begun. With regard to direction, much more data needs to be obtained. This paper described the situation where all directions were toward the same reference place (the Library). This was required for reasons of practicality in limiting the size of the experiment. A small data set concerning directions toward a selected number of different reference places has already been collected and is currently being analyzed. However, if a richer geometry of human conception of the spaces that we occupy is to emerge, more experimental evidence is needed.

7.2 Combinations of vague spatial relations

This paper highlights the importance of combinations of proximity and direction relations in achieving efficient qualitative spatial representation and reasoning. Combining proximity and direction can be theoretically justified if we make the reasonable assumption that people make proximity and direction judgments from a common mental representation. Combining nearness and leftness relations makes possible new inferences, such as presented in section 6.3, that cannot be achieved using leftness or nearness alone. Similarly, within the context of QVSs the combination of proximity and direction yields a smaller faithful minimal framework than either proximity or direction relations alone (section 6.4). This result provides experimental evidence that, as might be expected, a commonsense description of an environmental space using both proximity and direction relations can be more compact than one that uses only proximity or direction relations.

One motivation for the work reported here was a need to have quantitatively richer theories of vagueness in the case of spatial relationships. From preliminary work, proximity seemed a more natural concept to work with than distance. Proximity and distance are inverse concepts. However, “near” is clearly not the same as “not far,” and “far” is not equivalent to “not near,” so no simple relationship is obtained. Future work will aim to clarify the relationship between proximity and distance. Work from cognitive scientists on the inverse relationship between similarity and dissimilarity (e.g., conceptual distance, Gärdenfors, 2000) may throw some useful light here.

7.3 Granularity

The discussion in section 6.4 shows how three-valued proximity and direction relations can be interpreted as a QVS, which in turn provides a granulation of the environmental space. The ability to represent and reason about granularity in spatial information is a key goal for qualitative studies, and is required for most practical application domains. For example, users of a mobile location-aware navigation system require enough detail in their navigation instruction in order to be able to successfully reach their destination. Conversely, recent work has indicated that revealing too much detail about a user's location may be undesirable both for technical and privacy reasons (Duckham et al., 2003). Future work will need to address the use of QVSs as a formal basis for qualitative reasoning about granularity in such applications.

In section 6.4, a family of indiscernibility relations was introduced on the set of places. Two places are indiscernible with respect to one of these relations if their values at each predicate in the relation's framework are identical. Thus, each of these relations is an equivalence relation, and so imposes a partition (granularity) on the set of places. However, remembering that framework predicates are three-valued, it might be more natural to allow the indiscernibility relation to be three-valued as well. We are working on a generalized three-valued indiscernibility relation, where the relation is undecided in cases where there are "weak" disagreements in values (e.g., between $\top$ and $?$, or between $?$ and $\bot$). In this case the blocks of the generalized partition are egg-yolk regions (Cohn and Gotts, 1996); such generalized partitions have begun to be investigated by Ahlqvist et al. (2000).

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