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# GENERATORS FOR THE SPORADIC GROUP $Co_3$ AS A (2, 3, 7) GROUP

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## 1. Introduction

A $(2,3,7)$-group is a group generated by two elements, one an involution and the other of order 3, whose product has order 7. Known finite simple examples of such groups are $PSL(2,7)$, $PSL(2,p)$ where $p$ is prime and $p\equiv\pm1 \pmod 7$, $PSL(2,p^3)$ where $p$ is prime and $p\not\equiv 0, \pm1 \pmod 7$, groups of Ree type of order $q^3(q^3+1)(q-1)$ where $q=3^{2n+1}$ and $n>0$, the sporadic group of order $2^3\cdot3\cdot5\cdot7\cdot11\cdot19$ discovered by Janko, and the Hall–Janko–Wales group of order $2^7\cdot3^3\cdot5^2\cdot7$ [**4, 2**]. G. Higman in an unpublished paper has shown that every sufficiently large alternating group is a $(2,3,7)$-group. Here we show that the sporadic group $Co_3$ discovered by Conway [**1**] is a $(2,3,7)$-group.

I should like to thank Prof. D. Livingstone for suggesting to me the area of sporadic $(2,3,7)$-groups and for his supervision of [**5**], and Dr. R. List for some useful remarks related to the construction below.

## 2. Assumed properties of $Co_3$

Let $\Omega=\{1,\dots,276\}$, $\Omega'=\{1,\dots,100\}$, $\Omega''=\Omega\backslash\Omega'$, $\Gamma'=\{1,\dots,23\}$, $\Gamma''=\Omega\backslash\Gamma'$. If $\Delta\subseteq\Omega$ and $x$ acts on $\Omega$ as a permutation with $\Delta$ a union of cycles of $x$, then $x^\Delta$ will denote $x$ as a permutation of $\Delta$.

It is well known that $Co_3$ acts faithfully and doubly transitively as a group of permutations of $\Omega$, and that, viewed in this way, $Co_3$ has subgroups $H$ and $M$ with the following properties:

Property 1: $H$ is isomorphic to the Higman–Sims simple group. $H$ has orbits $\Omega'$ and $\Omega''$. $H_{[1]}$ is isomorphic to the Mathieu group $M_{22}$ and has among its orbits $\Gamma'\backslash\{1\}$ and $\Omega'\backslash\Gamma'$. The actions of $H_{[1]}$ on $\Gamma'\backslash\{1\}$ and $\Omega'\backslash\Gamma'$ correspond to the natural actions of $M_{22}$ on the 22 points and 77 blocks respectively of the Steiner system $S(3,6,22)$.

Property 2: $M$ is isomorphic to the Mathieu group $M_{23}$. $M$ has orbits $\Gamma'$ and $\Gamma''$ corresponding to the 23 points and 253 blocks respectively of the Steiner system $S(4,7,23)$ which is an extension of the $S(3,6,22)$ above.

## 3. Construction of the generators

Higman and Sims [**3**] have constructed generators $a$ and $b$ for $H\,|\,\Omega'$. R. List (unpublished) has calculated the corresponding representations of $a$ and $b$ as permutations of $\Omega''$. Concatenating the two representations gives us $a$ and $b$ as permutations of

$\Omega$ as follows:
(Commas are omitted for the sake of clarity.)

$a =$
(1)(23)(101)
(2 8 13 17 20 22 7)(3 9 14 18 21 6 12)
(4 10 15 19 5 11 16)(24 77 99 72 64 82 40)
(25 92 49 88 28 65 90)(26 41 70 98 91 38 75)
(27 55 43 78 86 87 45)(29 69 59 79 76 35 67)
(30 39 42 81 36 57 89)(31 93 62 44 73 71 50)
(32 53 85 60 51 96 83)(33 37 58 46 84 100 56)
(34 94 80 61 97 48 68)(47 95 66 74 52 54 63)
(102 103 104 105 106 107 108)(109 110 ........ 115)
(116 .................. 122) .....................
...............................................
(270 271 272 273 274 275 276)

$b =$
(2)(5)(16)(22)(27)(28)(29)(30)(33)(36)(47)(48)(73)(75)(79)
(80)(86)(88)(93)(94)(114)(138)(147)(153)(155)(160)(183)(189)(198)
(209)(220)(238)(251)(266)(273)(274)
(1 35)(3 81)(4 92)(6 60)(7 59)(8 46)(9 70)(10 91)
(11 18)(12 66)(13 55)(14 85)(15 90)(17 53)(19 45)(20 68)
(21 69)(23 84)(24 34)(25 31)(26 32)(37 39)(38 42)(40 41)
(43 44)(49 64)(50 63)(51 52)(54 95)(56 96)(57 100)(58 97)
(61 62)(65 82)(67 83)(71 98)(72 99)(74 77)(76 78)(87 89)
(101 102)(103 109)(104 116)(105 123)(106 130)(107 137)(108 144)(110 125)
(111 151)(112 158)(113 165)(115 172)(117 149)(118 179)(119 145)(120 186)
(121 193)(122 200)(124 207)(126 208)(127 132)(128 214)(129 221)(131 187)
(133 228)(134 177)(135 146)(136 148)(139 235)(140 229)(141 175)(142 154)
(143 216)(150 232)(152 159)(156 197)(157 230)(161 242)(162 249)(163 227)
(164 192)(166 188)(167 233)(168 256)(169 218)(170 213)(171 263)(173 248)
(174 265)(176 181)(178 264)(180 259)(182 206)(184 267)(185 204)(190 268)
(191 239)(194 224)(195 205)(196 219)(199 270)(201 262)(202 217)(203 241)
(210 260)(211 240)(212 257)(215 254)(222 247)(223 237)(225 250)(226 275)
(231 244)(234 271)(236 258)(243 261)(245 276)(246 269)(252 272)(253 255)

Let $x = ba^{-1}ba^2ba^3$ and $y = xa^3bx^{-1}$. Then $y^\Gamma = (1)(2, 8, 10, 17, 19, 6, 18, 15, 11, 7, 12)(3, 23, 22, 9, 16, 14, 20, 4, 21, 13, 5)$ and $\langle a, y\rangle \simeq M_{22}$. We now extend to $M_{23}$ on $\Gamma'$ and calculate the action on the remaining points of $\Omega$.

Let $c^\Gamma = (1, 2, 20, 23, 9, 19, 12, 13, 18, 3, 21, 8, 10, 4, 6, 14, 11, 22, 7, 17, 16, 5, 15)$. Allowing $\langle a^{\Gamma'}, c^{\Gamma'}, y^{\Gamma'}\rangle$ to act on the set $\{3, 6, 9, 12, 14, 18, 21\}$, we generate a Steiner system $S(4, 7, 23)$. Thus $\langle a^{\Gamma'}, c^{\Gamma'}, y^{\Gamma'}\rangle$ is isomorphic to $M_{23}$.

We extend $c^{\Gamma'}$ to $c \mid \Omega$ in the following way. Knowing the action of $\langle a, y\rangle$ on $\Gamma''$ and remembering Property 2, we place $\Gamma''$ in one-one correspondence with the set of blocks of the above $S(4, 7, 23)$ so that the action of $\langle a, y\rangle$ on the set of points $\Gamma'$ is consistent with its action on the blocks. We can now find the action of $c$ on $\Gamma''$ by calculating the action of $c^{\Gamma'}$ on the blocks of $S(4, 7, 23)$.

Since $a$, $b$ and $c$ have been constructed so that Properties 1 and 2 hold, and since the construction may be made in essentially only one way, $\langle a, b, c\rangle < Co_3$.

Let $d = (ab)^2(ba^4c)^3(ab)^{-2}$ and $e = c^9(ba^4c)^2c^{-9}$.

$d =$
(13)(80)(103)(105)(124)(147)(161)(162)(187)(251)(256)(266)
(1 23)(2 174)(3 254)(4 216)(5 98)(6 246)(7 191)(8 230)
(9 192)(10 253)(11 234)(12 131)(14 264)(15 172)(16 85)(17 144)
(18 155)(19 70)(20 186)(21 26)(22 188)(24 168)(25 81)(27 71)
(28 114)(29 57)(30 58)(31 77)(32 235)(33 272)(34 126)(35 169)
(36 37)(38 146)(39 99)(40 111)(41 143)(42 69)(43 88)(44 198)
(45 49)(46 65)(47 89)(48 158)(50 195)(51 217)(52 110)(53 138)
(54 193)(55 62)(56 181)(59 82)(60 239)(61 241)(63 225)(64 180)
(66 222)(67 170)(68 154)(72 226)(73 197)(74 223)(75 267)(76 118)
(78 227)(79 270)(83 236)(84 86)(87 221)(90 100)(91 244)(92 121)
(93 115)(94 159)(95 171)(96 135)(97 129)(101 249)(102 240)(104 231)
(106 248)(107 250)(108 252)(109 211)(112 209)(113 179)(116 117)(119 196)
(120 274)(122 165)(123 247)(125 149)(127 269)(128 139)(130 164)(132 157)
(133 151)(134 190)(136 206)(137 205)(140 237)(141 238)(142 220)(145 229)
(148 160)(150 265)(152 177)(153 268)(156 178)(163 260)(166 271)(167 213)
(173 201)(175 245)(176 214)(182 184)(183 212)(185 224)(189 218)(194 210)
(199 276)(200 204)(202 255)(203 242)(207 208)(215 219)(228 263)(232 257)
(233 258)(243 261)(259 262)(273 275)

$e =$
(1 121 184)(2 247 265)(3 112 172)(4 145 199)(5 12 110)(6 242 15)
(7 123 232)(8 229 26)(9 233 194)(10 72 143)(11 85 165)(13 268 212)
(14 236 222)(16 191 269)(17 50 73)(18 37 79)(19 155 214)(20 56 47)
(21 180 215)(22 127 271)(23 203 32)(24 108 104)(25 263 94)(27 149 138)
(28 176 213)(29 162 125)(30 192 58)(31 257 189)(33 273 246)(34 70 230)
(35 88 43)(36 53 136)(38 252 64)(39 206 197)(40 193 109)(41 237 200)
(42 274 140)(44 154 83)(45 251 84)(46 201 67)(48 202 256)(49 241 270)
(51 218 141)(52 181 147)(54 98 150)(55 126 157)(57 167 144)(59 92 170)
(60 178 132)(61 245 195)(62 185 71)(63 196 225)(65 78 119)(66 211 234)
(68 105 248)(69 97 159)(74 111 137)(75 182 91)(76 238 239)(77 231 275)
(80 115 217)(81 266 187)(82 164 128)(86 99 153)(87 171 139)(89 204 133)
(90 208 160)(93 107 101)(95 226 96)(100 209 177)(102 166 272)(103 267 254)
(106 255 205)(113 261 227)(114 259 116)(117 163 276)(118 219 134)
(120 130 260)(122 264 169)(124 175 183)(129 221 228)(131 207 253)
(135 142 244)(146 179 235)(148 223 240)(151 262 224)(152 161 220)
(156 174 186)(158 198 188)(168 249 210)(173 243 216)(190 258 250)

It is straightforward to calculate that

$$d^2 = e^3 = (de)^7 = 1.$$

It may further be checked that $\langle d, e\rangle$ is 2-transitive on $\Omega$. Since $Co_3$ has no proper subgroups which are 2-transitive on $\Omega$, $\langle d, e\rangle$ is isomorphic to $Co_3$ and $Co_3$ is a $(2, 3, 7)$-group.

## 4. Remarks

In fact we have been able to show that $Co_3$ has exactly 12 non-isomorphic presentations as a $(2, 3, 7)$-group. Using the notation $(l, m, n; q)$ to mean the group with presentation

$$\langle x, y: x^l = y^m = (xy)^n = (x^{-1}y^{-1}xy)^q = 1\rangle,$$

$Co_3$ is a factor group of $(2, 3, 7; 14)$ in four distinct ways (up to presentaion isomorphism), $(2, 3, 7; 24)$ in six distinct ways and $(2, 3, 7; 30)$ in two distinct ways. The details are omitted but may be found in [**5**].

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